Math 5800

12/6/21

HW problem discussion

After class on 12/1 (Weds) we talked about HW 8 #1 which is about showing a function f is not in L^1.

This discussion is on the class recording at the very end of 12/1. Check it out. [Fast forward to $\approx$ 1:03:39 in recording]

Note: In the recording I wrote $\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{k+1}$ under a sum which is wrong. It should be $\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1}$. I fixed this in HW solutions.

Final exam

- Weds Dec 15
- opens at 5am on Weds 12/15 and closes at 12pm noon on Thursday 12/16.
- You will get a 3 hr window to take the exam

- covers:
    - Test 1 material
    - Test 2 material
    - HW 8
    - HW 9

- I emailed out a more thourough study guide

Theorem: Let $E, F \in M$

[That is, E and F are measurable sets.]

Then:

1. $\emptyset \in M$ and $\mu(\emptyset) = 0$
2. If $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
3. We have that
$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$$
4. Thus,
$$\mu(E \cup F) \leq \mu(E) + \mu(F)$$
and if $E \cap F = \emptyset$ then
$$\mu(E \cap F) = \mu(E) + \mu(F).$$
5. If $F \subseteq E$ and $\underbrace{\mu(F) < \infty}_{F \text{ is integrable}}$, then
$$\mu(E - F) = \mu(E) - \mu(F)$$

(6) If $E_1, E_2, \ldots, E_n$ are measurable, then

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} \mu(E_k)$$

$\boxed{\text{pg } 4}$

Moreover, if $E_1, E_2, \ldots, E_n$ are mutually disjoint [ie $E_i \cap E_j = \phi$ if $i \neq j$]

then

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k)$$

(7) If $E_1, E_2, \ldots$ is a sequence of measurable sets then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

If $E_1, E_2, \ldots$ are mutually disjoint [ie $E_i \cap E_j = \phi$ if $i \neq j$] then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

proof:

① $\underbrace{\chi_\phi}_{\text{zero function}} = \chi_{(1,1)} \in L^1 \subseteq \underbrace{\tilde{M}}_{\text{measurable functions}}$

So, ϕ is measurable.

And,

$\mu(\phi) = \int \chi_\phi = \int \chi_{(1,1)} = 1 - 1 = 0$

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② Suppose $E \subseteq F$.

case 1: Suppose F is integrable,

ie $\chi_F \in L^1$.

By the lemma from Weds, since $E \subseteq F$ and F is integrable, we know that E is integrable.

Thus, $\chi_E \in L^1$.

Because $E \subseteq F$, by HW, we know $\chi_E(x) \leq \chi_F(x)$ for all x.

Since $\chi_E, \chi_F \in L^1$ and $\chi_E \leq \chi_F$, we know

$$\mu(E) = \int \chi_E \leq \int \chi_F = \mu(F)$$

<u>case 2:</u> Suppose F is measurable, but not integrable.

Then, $\mu(F) = \infty$.

Since $\mu(E)$ is finite or $\mu(E) = \infty$ we know

$$\underbrace{\mu(E)}_{\#\text{ or } \infty} \leq \underbrace{\mu(F)}_{\infty}.$$

(3) Let E and F be measurable.

We must show

$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$$

First of all, note that by Weds since E and F are measurable we know $E \cup F$ and $E \cap F$ are measurable.

Also, by the previous lemma, if $E \cup F$ is integrable then since $E \subseteq E \cup F$ and $F \subseteq E \cup F$ we would have E and F integrable.

$\boxed{\text{Case 1: Suppose } E \text{ is not integrable}}$

Then, $E \cup F$ cannot be integrable.

Thus, $\mu(E) = \infty$ and $\mu(E \cup F) = \infty$.

This makes

$$\underbrace{\mu(E \cup F)}_{\infty} + \mu(E \cap F) = \underbrace{\mu(E)}_{\infty} + \mu(F)$$

true.

$\boxed{\text{Case 2: Suppose } F \text{ is not integrable}}$

Same proof as case 1, just interchange E and F.

$\boxed{\text{Case 3: Suppose } E \text{ and } F \text{ are both integrable}}$

Then, $\chi_E \in L^1$ and $\chi_F \in L^1$.

Thus, by HW 9 #5(c),

$$\chi_{E\cup F} = \max\{\chi_E, \chi_F\} \in L^1$$

So, $E\cup F$ is integrable.

Because $E\cap F \subseteq E$ and E is integrable, by the lemma $E\cap F$ is integrable.

By HW we know

$$\chi_{E\cup F} = \chi_E + \chi_F - \chi_{E\cap F}$$

Thus, $\mu(E\cup F) + \mu(E\cap F)$

$$= \int \chi_{E\cup F} + \int \chi_{E\cap F}$$

$$= \int (\chi_E + \chi_F - \chi_{E\cap F}) + \int \chi_{E\cap F}$$

$$= \int \chi_E + \int \chi_F - \int \chi_{E\cap F} + \int \chi_{E\cap F}$$

$$= \int \chi_E + \int \chi_F = \mu(E) + \mu(F).$$

(4) Follows from part 1 and part 3

(5) You can try.

(6) Let $E_1, E_2, \ldots, E_n$ be measurable.

We want to show

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} \mu(E_k) \qquad (*)$$

and we get $=$ if the sets are mutually disjoint.

If any of $E_1, E_2, \ldots, E_n$ are not integrable, then $\bigcup_{k=1}^{n} E_k$ will not be integrable.

Weds lemma

In this case both sides of $(*)$ will be ∞ and thus equal.

Thus we can assume all of $E_1, E_2, \ldots, E_n$ are integrable.

This will imply that χ_E is integrable where $E = E_1 \cup \cdots \cup E_n$

[use $\chi_E = \max\{\chi_{E_1}, \chi_{E_2}, \ldots, \chi_{E_n}\}$]

Since $\chi_E(x) \leq \sum_{k=1}^{n} \chi_{E_k}(x)$ for all x. [Hw 4 #3]

we have that

$$\mu(E) = \int \chi_E \leq \sum_{k=1}^{n} \int \chi_{E_k}$$

$$= \sum_{k=1}^{n} \mu(E_k)$$

If $E_1, E_2, \ldots, E_n$ are mutually disjoint then by Hw 4 #4, $\chi_E = \sum_{k=1}^{n} \chi_{E_k}$

Thus,

$$\mu(E) = \int \chi_E = \sum_{k=1}^{n} \int \chi_{E_k} = \sum_{k=1}^{n} \mu(E_k).$$

(7) If any of the $E_1, E_2, E_3, \ldots$ are not integrable, then $\bigcup_{k=1}^{\infty} E_k$ is not integrable and $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \infty$ and $\sum_{k=1}^{\infty} \mu(E_k) = \infty$ making $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$ true.

So we may assume each of $E_1, E_2, \ldots$ are integrable.

Define $A = \bigcup_{k=1}^{\infty} E_k$.

Define $A_n = \bigcup_{k=1}^{n} E_k = E_1 \cup E_2 \cup \ldots \cup E_n$ for each $n \geq 1$.

Then A_n is integrable

Then, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$

So $\chi_{A_1} \leq \chi_{A_2} \leq \chi_{A_3} \leq \ldots$

Thus, $(\chi_{A_n})_{n=1}^{\infty}$ is a non-decreasing sequence.

We may assume that $\left(\int \chi_{A_n}\right)_{n=1}^{\infty}$ is a bounded sequence.

Why?

Suppose not, that is suppose $\lim_{n \to \infty} \int \chi_{A_n} = \infty$

Since

$$\int \chi_{A_n} = \int \chi_{E_1 \cup \cdots \cup E_n} \le \sum_{k=1}^{n} \int \chi_{E_k}$$

$$= \sum_{k=1}^{n} \mu(E_k).$$

This would imply $\sum_{k=1}^{\infty} \mu(E_k) = \infty$

making the theorem true.

Thus we have a non-decreasing sequence $(\chi_{A_n})_{n=1}^{\infty}$ of L^1 functions with $(\int \chi_{A_n})_{n=1}^{\infty}$ bounded

and $\chi_{A_n} \to \chi_A$ for all $x \in \mathbb{R}$.

Thus by the monotone convergence Thm $\chi_A \in L^1$

and $\lim_{n\to\infty} \int \chi_{A_n} = \int \chi_A$.

So,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu(A) = \int \chi_A =$$

$$= \lim_{n\to\infty} \int \chi_{A_n}$$

$$= \lim_{n\to\infty} \int \chi_{E_1 \cup \cdots E_n} \leq \lim_{n\to\infty} \sum_{k=1}^{n} \int \chi_{E_k}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E_k)$$

$$= \sum_{k=1}^{\infty} \mu(E_k).$$

This shows $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$.

If the sets are disjoint then

$$\chi_{E_1 \cup \cdots \cup E_n} = \sum_{k=1}^{n} \chi_{E_k}$$

and we will get equality above giving

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$