Math 5800

8/23/21

- I'll use your calstatela.edu email to mass email the class announcements. If you want me to use a different email just let me know.

- Tests are done on canvas. No class on the test day. The test will appear on canvas at 6am on the test day and dissapear at noon the following day.
[Ex: Mon 6am – Tues noon]
During that window you pick a 2.5 hour time to take the test. When you open the test canvas will time you. When done scan and upload as a pdf.

Topic 1 – Countable and Uncountable sets

<u>Def:</u> We say that a set S is <u>countably infinite</u> if S is infinite and the elements of S can be arranged in a sequence without repeats, that is S is of the form

$$S = \{ s_1, s_2, s_3, s_4, \ldots \}$$

Another def is S is countably infinite if there exists a bijection $f: \mathbb{N} \to S$. Then,

$$S = \{ f(1), f(2), f(3), f(4), \ldots \}$$

<u>Def:</u> We say that S is <u>countable</u> if S is finite or S is countably infinite.

<u>Ex:</u> Some countable sets:

$S = \{5, 10, \pi, 3\}$ ← finite

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots\}$

countably infinite

<u>Theorem:</u> Let A and B be countable. Then $A \cup B$ is countable.

<u>proof:</u>

<u>case 1:</u> If A and B are both finite then $A \cup B$ is finite and so its countable.

<u>case 2:</u> Suppose $A = \{a_1, a_2, \ldots, a_n\}$ is finite and

$$B = \{b_1, b_2, b_3, b_4, \ldots\}$$

is countably infinite.

Then

$$A \cup B = \{a_1, a_2, \ldots, a_n, b_1, b_2, b_3, b_4, \ldots\}$$

is an enumeration of $A \cup B$. Skip any duplicates when you enumerate.

Case 3: If B is finite and A is countably infinite do the same thing as case 2.

Case 4: Suppose

$$A = \{a_1, a_2, a_3, a_4, \ldots\}$$

and

$$B = \{b_1, b_2, b_3, b_4, \ldots\}$$

are both countably infinite and the above are enumerations without duplicates.

Then

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\}.$$

Skip any repeats.

So, $A \cup B$ is countable. $\blacksquare$

<u>Theorem:</u> The rational numbers $\mathbb{Q}$ is countable.

<u>proof:</u> Let $\mathbb{Q}_+ = \{r \mid r > 0, r \in \mathbb{Q}\}$ and $\mathbb{Q}_- = \{r \mid r < 0, r \in \mathbb{Q}\}$.

Then, $\mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\}$.

Let's show that $\mathbb{Q}_+$ is countable.

Consider the following way to arrange $\mathbb{Q}_+$ where we cross out any elements not in lowest form and circle the elements of $\mathbb{Q}_+$.

$$\begin{array}{cccccc}
\frac{1}{1}, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & \cdots \\
\frac{2}{1}, & \cancel{\frac{2}{2}}, & \frac{2}{3}, & \cancel{\frac{2}{4}}, & \frac{2}{5}, & \cdots \\
\frac{3}{1}, & \frac{3}{2}, & \cancel{\frac{3}{3}}, & \frac{3}{4}, & \frac{3}{5}, & \cdots \\
\frac{4}{1}, & \cancel{\frac{4}{2}}, & \frac{4}{3}, & \frac{4}{4}, & \frac{4}{5}, & \cdots \\
\frac{5}{1}, & \frac{5}{2}, & \frac{5}{3}, & \frac{5}{4}, & \frac{5}{5}, & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Continuing in this way forever yields a sequence which enumerates $\mathbb{Q}_+$ without repeats. That is,

$$\mathbb{Q}_+ = \left\{ 1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4, 5, 1/5, \cdots \right\}$$

So, $\mathbb{Q}_+$ is countable.

Similarly $\mathbb{Q}_-$ is countable.

Also, $\{0\}$ is countable.

Thus, by the previous theorem

$$\mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\}$$

is countable. $\blacksquare$

<u>Theorem:</u> The union of a countable number of countable sets is countable.

<u>proof:</u>

<u>case 1:</u> Suppose we have a finite number of countable sets $S_1, S_2, S_3, \ldots, S_n$.

Then by iterating the theorem on page 4 we have that

$$S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_n$$

is countable.

<u>case 2:</u> Suppose we have a countably infinite number of countable sets

$$S_1, S_2, S_3, S_4, \ldots$$

Suppose

$$S_i = \{ s_{i1}, s_{i2}, s_{i3}, \ldots \}$$

Arrange the sets as follows and enumerate like with $\mathbb{Q}_+$:

$$\begin{array}{cccccccl}
s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & \cdots & \leftarrow & S_1 \\
s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & \cdots & \leftarrow & S_2 \\
s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & \cdots & \leftarrow & S_3 \\
s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & \cdots & \leftarrow & S_4 \\
s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & \cdots & \leftarrow & S_5 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots
\end{array}$$

List out

$$\bigcup_{i=1}^{\infty} S_i = \{ s_{11}, s_{12}, s_{21}, s_{31}, s_{22}, s_{13}, \ldots \}$$

as above, skipping any repeats, to show that $\bigcup_{i=1}^{\infty} S_i$ is countable. $\blacksquare$