Math 5800

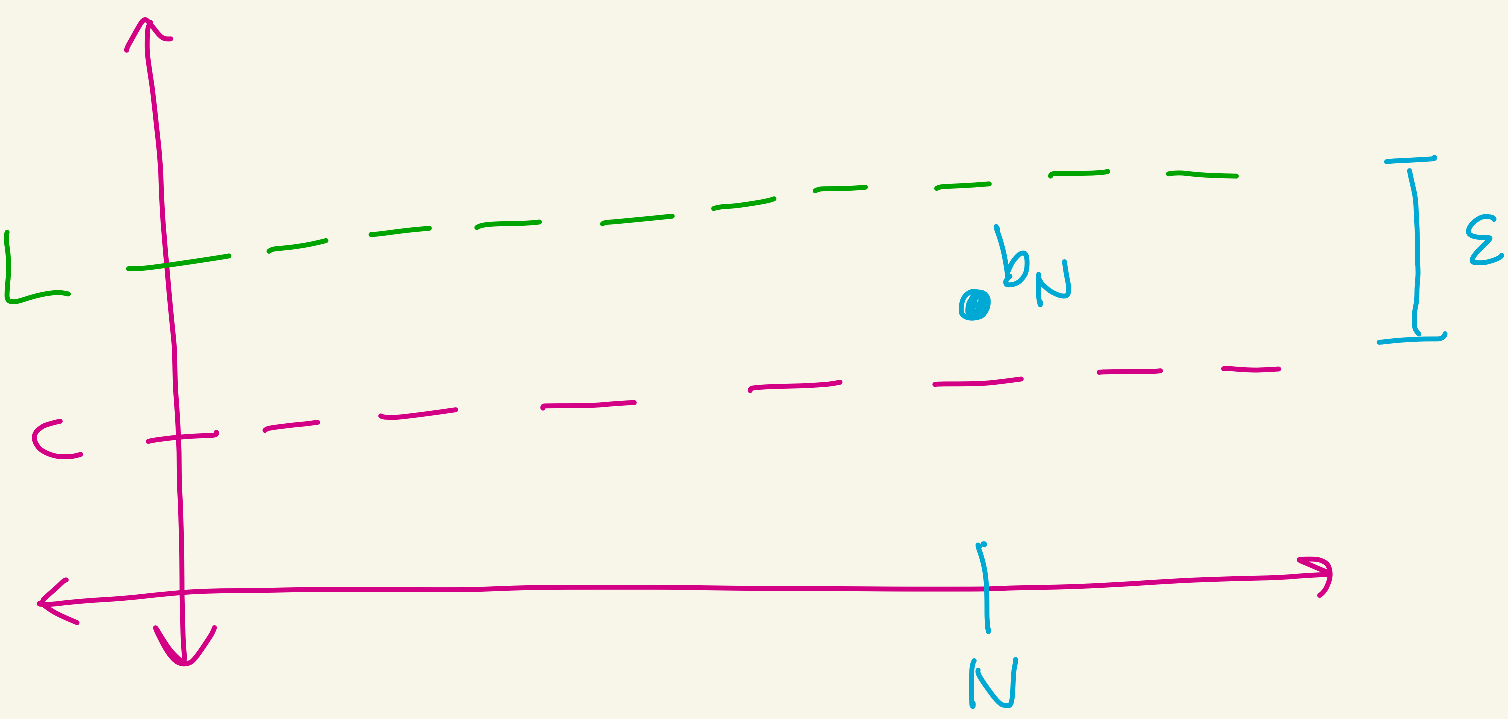
9/1/21

**Lemma:** Let $(b_n)_{n=1}^{\infty}$ be a convergent sequence of real numbers.

If there exists $c \in \mathbb{R}$ where $b_n < c$ for all $n \geq 1$, then $\lim\limits_{n\to\infty} b_n \leq c$.

**proof:** Let $L = \lim\limits_{n\to\infty} b_n$.

Suppose $L > c$.



Let $\varepsilon = L - c > 0$.

Since $b_n \to L$, there exists $N > 0$ where if $n \geq N$ we have $\underbrace{L - \varepsilon < b_n < L + \varepsilon}$

Same as: $|b_n - L| < \varepsilon$

So for example, $\underbrace{L - \varepsilon < b_N < L + \varepsilon}$

So, $L - \underbrace{(L - c)}_{\varepsilon} < b_N$

Thus, $c < b_N$

This can't happen.

Thus, $L \leq c$. $\blacksquare$

Theorem:

Let $A_1, A_2, A_3, \ldots$ be a countably infinite number of measure zero subsets of $\mathbb{R}$.

Then, $A = \bigcup_{k=1}^{\infty} A_k$ has measure zero.

proof is from Weir pg 19

proof: Let $\varepsilon > 0$.

Since $A_1$ has measure zero there exists bounded open intervals

$I_{11}, I_{12}, I_{13}, \ldots$

where $A_1 \subseteq \bigcup_{j=1}^{\infty} I_{1j}$ and $\sum_{j=1}^{\infty} \ell(I_{1j}) \leq \frac{\varepsilon}{2}$

Since $A_2$ has measure zero there exists bounded open intervals

$I_{21}, I_{22}, I_{23}, \ldots$

where $A_2 \subseteq \bigcup_{j=1}^{\infty} I_{2j}$ and $\sum_{j=1}^{\infty} \ell(I_{2j}) \leq \frac{\varepsilon}{2^2}$

In general, for each $k \geq 1$, since $A_k$ has measure zero there exists bounded open intervals

$$I_{k1}, I_{k2}, I_{k3}, \ldots$$

where $A_k \subseteq \bigcup_{j=1}^{\infty} I_{kj}$

and $\sum_{j=1}^{\infty} \ell(I_{kj}) \leq \frac{\varepsilon}{2^k}$

Since $A = \bigcup_{k=1}^{\infty} A_k$ we know that

$$A \subseteq \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} I_{kj} .$$

We are gonna rearrange the ordering of the $I_{kj}$'s now.

Arrange the open intervals as follows:

$$\begin{array}{ccccc} I_{11} & I_{12} & I_{13} & \cdots & \leftarrow A_1 \\ I_{21} & I_{22} & I_{23} & \cdots & \leftarrow A_2 \\ I_{31} & I_{32} & I_{33} & \cdots & \leftarrow A_3 \\ \vdots & \vdots & \vdots & & \end{array}$$

We can thus order all these intervals as follows:

$$\underbrace{I_{11},}_{\text{subscripts add to 2}} \underbrace{I_{12}, I_{21},}_{\text{subscripts add to 3}} \underbrace{I_{31}, I_{22}, I_{13},}_{\text{subscripts add to 4}} \cdots \quad (*)$$

As we said before

$$A \subseteq I_{11} \cup I_{12} \cup I_{21} \cup I_{31} \cup \cdots$$

We want to sum the lengths of the intervals in $(*)$ and show the infinite sum is $\leq \varepsilon$.

Suppose we look at the sum of the lengths of the first $n$ terms in $(*)$.

For example, if we calculate the sum of the lengths of the first $n=5$ terms in $(*)$ we have:

$$\ell(I_{11})+\ell(I_{12})+\ell(I_{21})+\ell(I_{31})+\ell(I_{22})$$

$$=\underbrace{\ell(I_{11})+\ell(I_{12})}_{A_1}+\underbrace{\ell(I_{21})+\ell(I_{22})}_{A_2}+\underbrace{\ell(I_{31})}_{A_3}$$

$$\leq \sum_{j=1}^{\infty}\ell(I_{1j})+\sum_{j=1}^{\infty}\ell(I_{2j})+\sum_{j=1}^{\infty}\ell(I_{3j})$$

$$\boxed{\text{Corollary last class}} \quad \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}+\frac{\varepsilon}{2^3}$$

In general, if we look at the first n terms of (*) we will see that these sets are at most amongst the sets $I_{kj}$ that cover $A_1, A_2, \ldots, A_n$.

[Because we will get to at most the n-th row of our diagram]

Thus, the sum of the lengths of the first n terms of (*) is less than or equal to

$$\frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \cdots + \frac{\varepsilon}{2^n}$$

$$= \frac{\varepsilon}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} \cdots + \frac{1}{2^{n-1}}\right]$$

$$= \frac{\varepsilon}{2}\left[\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}\right] = \varepsilon\left(1 - \frac{1}{2^n}\right) < \varepsilon$$

$$1 + x + x^2 + \cdots + x^m = \frac{1 - x^{m+1}}{1 - x}, \quad x \neq 1$$

Thus, the sequence of partial sums corresponding to adding up the lengths of the sets from $(*)$

$$s_1 = \ell(I_{11}) \quad \leftarrow \boxed{n=1}$$

$$s_2 = \ell(I_{11}) + \ell(I_{12}) \quad \leftarrow \boxed{n=2}$$

$$s_3 = \ell(I_{11}) + \ell(I_{12}) + \ell(I_{21}) \quad \leftarrow \boxed{n=3}$$

$$\vdots \qquad \vdots$$

is a non-decreasing sequence of non-negative real #s whose terms are always $< \varepsilon$.

By the monotone convergence theorem $\lim_{n\to\infty} s_n$ exists.

Since $s_n < \varepsilon$ for all $n \geq 1$, we must have $\lim_{n\to\infty} s_n \leq \varepsilon$. lemma

Thus, A has measure zero. $\blacksquare$

<u>Corollary:</u> Let $A_1, A_2, \ldots, A_n$ be a finite number of sets of measure zero. Then,

$$A_1 \cup A_2 \cup \cdots \cup A_n$$

has measure zero.

<u>proof:</u>

Define $A_{n+1} = \emptyset$, $A_{n+2} = \emptyset, \ldots$

Ie, $A_k = \emptyset$ for $k \geq n+1$.

$\emptyset$ has measure zero.

And, $\bigcup_{k=1}^{n} A_k =$

$$= A_1 \cup A_2 \cup \cdots \cup A_n$$

$$= A_1 \cup A_2 \cup \cdots \cup A_n \cup \underbrace{A_{n+1}}_{\emptyset} \cup \underbrace{A_{n+2}}_{\emptyset} \cup \cdots$$

$$= \bigcup_{k=1}^{\infty} A_k$$

Every set in the union has measure zero.

Hence, $\bigcup_{k=1}^{\infty} A_k$ has measure zero.

So, $\bigcup_{k=1}^{n} A_n$ has measure zero. $\blacksquare$