Math 5800

9/13/21

$\boxed{\text{Typos}}$ – FIXED ONLINE

9/1/21 notes pg 8

A_{11}, A_{12}, A_{21}
should have been
I_{11}, I_{12}, I_{21}

I emailed everyone about this one.

9/8/21 notes pg 11

$\{x \in A \mid f(x) \neq g(x)\}$
$= \{1, 2\}$

should be $\{0, 1\}$

HW 3 - Measure Zero

Problem 5(b) had a typo in def of g
In last line had "$x \notin \mathbb{Z}$" should be "$x \in \mathbb{Z}$"

HW 2 Review

Solution for 4 had $|a_n + L|$
and should have been $|a_n - L|$

Test 1

Starts with

HW 3 - Measure Zero

HW 4 - Step Functions

Probably just those two HW.

Topic 4 – Step Functions

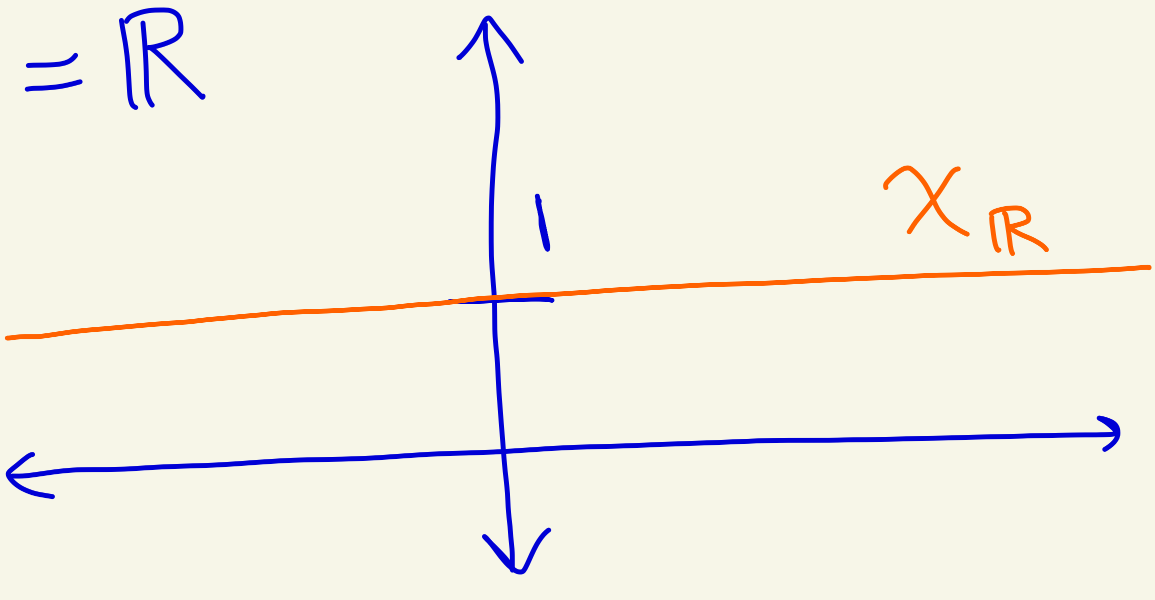
Def: Let $S \subseteq \mathbb{R}$.

The characteristic function on S is

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

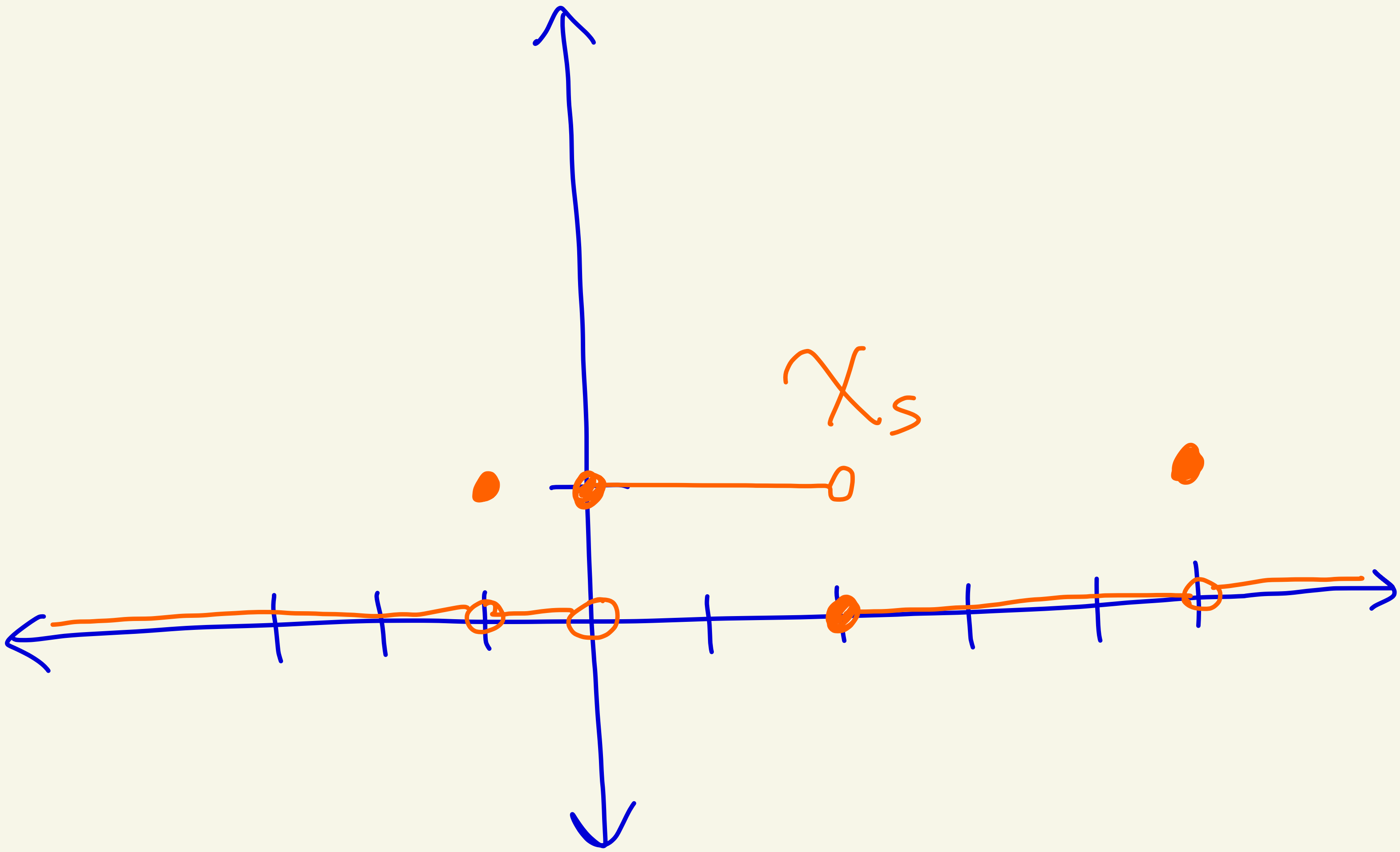
chi

Ex: $S = \mathbb{R}$



Ex:

$$S = [0,2) \cup \{-1, 5\}$$



Def: [Def 1.3.2 in WJ book]

A step function on $\mathbb{R}$ is a function f of the form

$$\boxed{f = \sum_{j=1}^{n} c_j \chi_{I_j}}$$

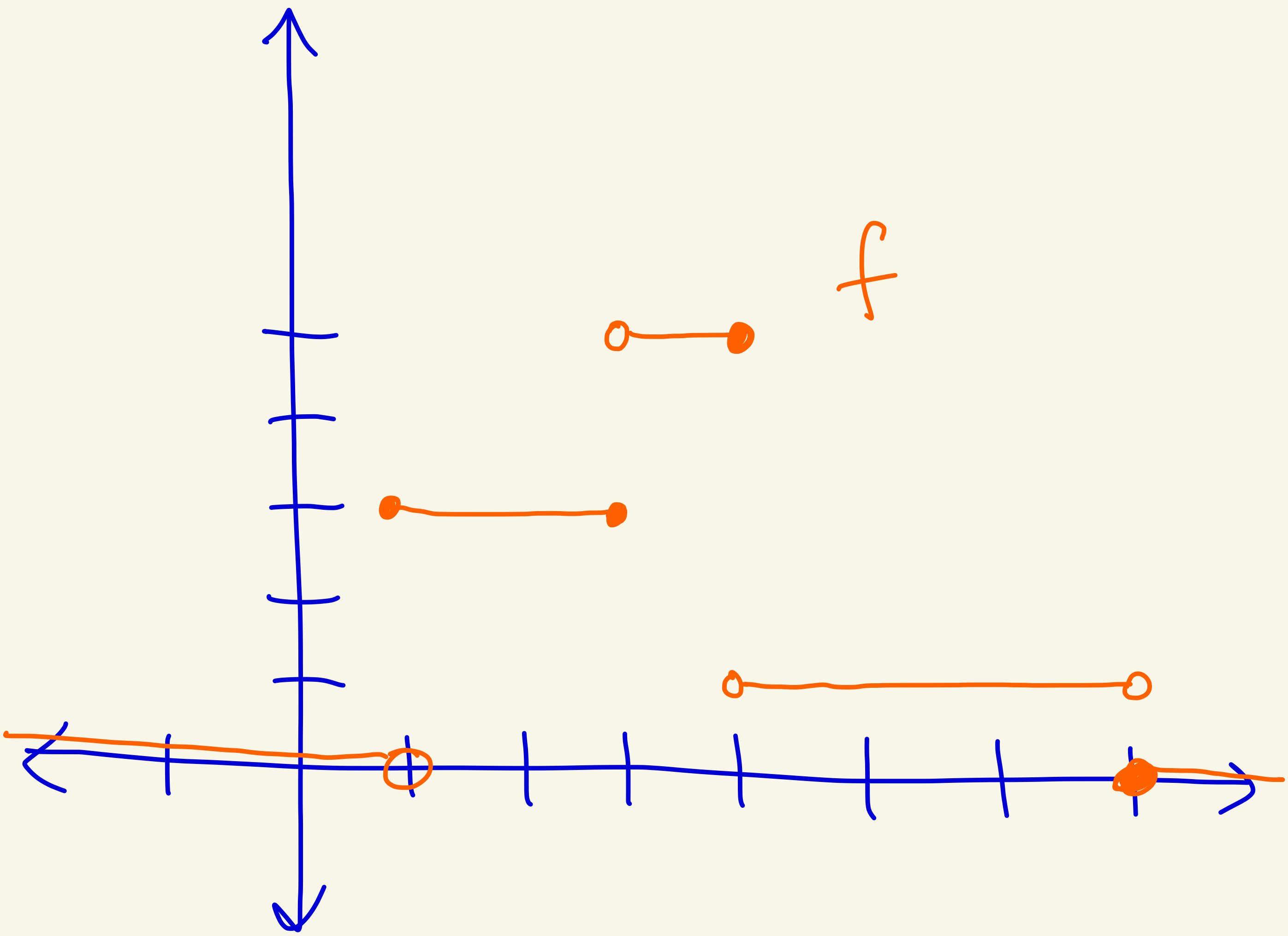
where each $c_j \in \mathbb{R}$ and each I_j is a bounded interval.

ex: $[0,2)$
$[1,1] = \{1\}$
$(1,1) = \emptyset$

$$\boxed{f(x) = \sum_{j=1}^{n} c_j \cdot \chi_{I_j}(x)}$$

__Ex:__ [Ex 1.3.1 from WJ]

$$f = 3 \cdot \chi_{[1,3]} + 5 \cdot \chi_{(3,4]} + 1 \cdot \chi_{(4,7)}$$



Def: [Def 1.3.3 in WJ] pg 7

The Lebesgue integral of the step function

$$f = \sum_{j=1}^{n} c_j \chi_{I_j}$$

is defined to be

$$\int f = \sum_{j=1}^{n} c_j \cdot \ell(I_j)$$

also be written as $\int_{\mathbb{R}} f$ or $\int_{-\infty}^{\infty} f(x)\,dx$

Ex:

$$f = 3 \cdot \chi_{[1,3]} + 5 \cdot \chi_{(3,4]} + 1 \cdot \chi_{(4,7)}$$

$$\int f = 3 \cdot \ell([1,3]) + 5 \cdot \ell((3,4]) + 1 \cdot \ell((4,7))$$

$$= 3(2) + 5(1) + 1 \cdot (3) = 14$$

Ex: Let

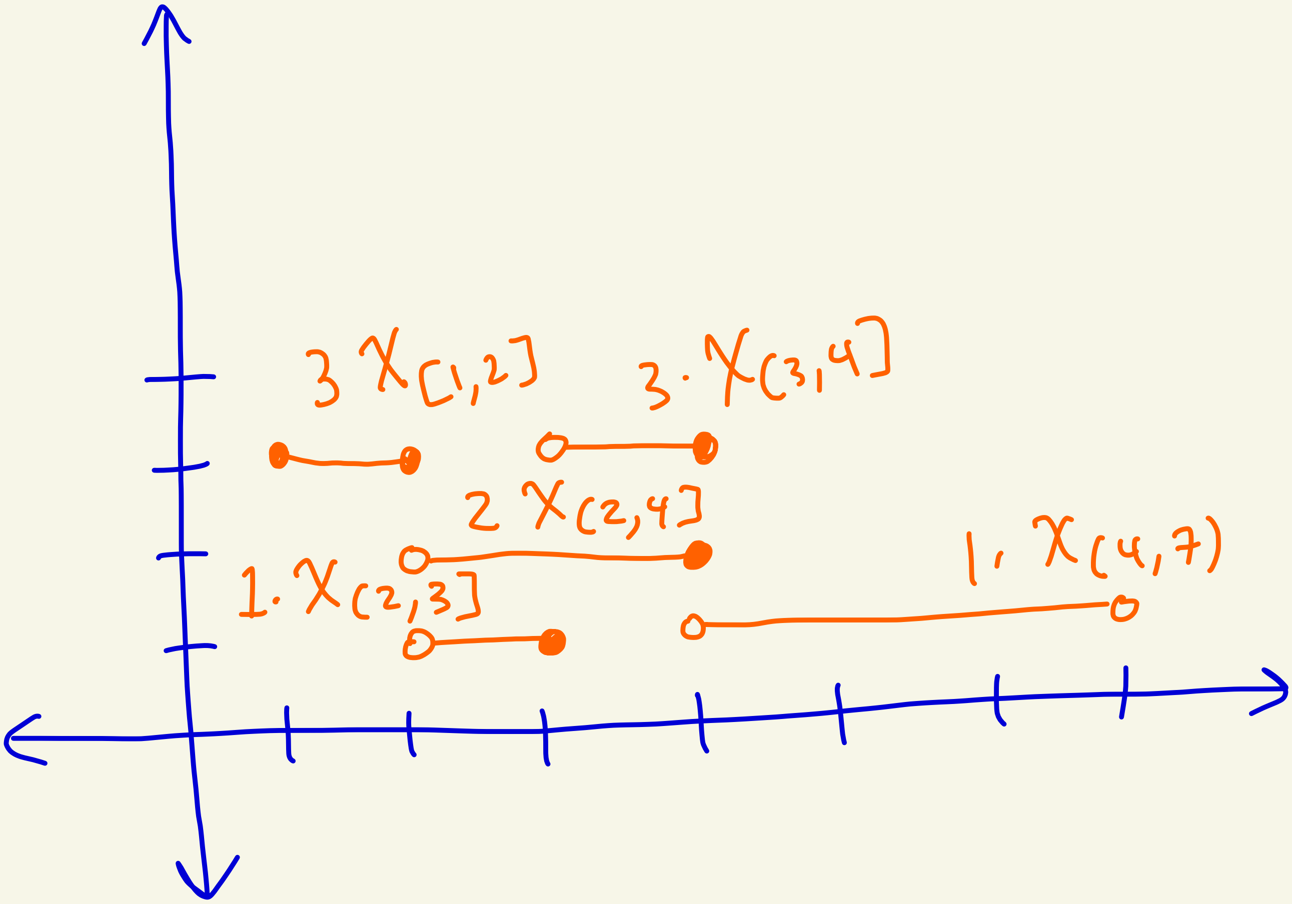
$$g = 3\cdot\chi_{[1,2]} + 1\cdot\chi_{(2,3]} + 2\cdot\chi_{(2,4]} + 3\cdot\chi_{(3,4]} + 1\cdot\chi_{(4,7)}$$

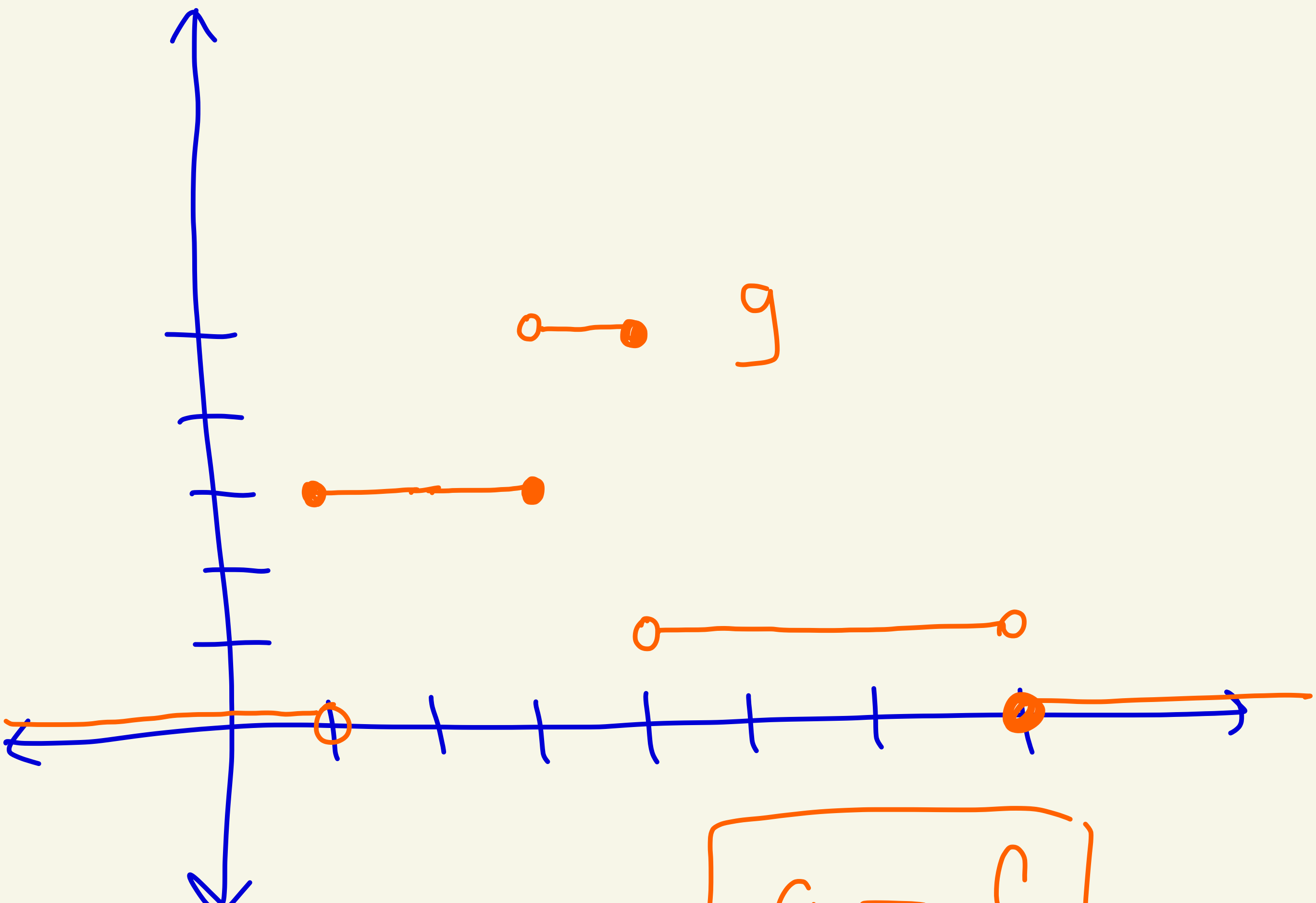
Then,

$$\int g = 3(1) + 1\cdot(1) + 2\cdot(2) + 3\cdot(1) + 1\cdot(3) = 14$$

Guess what?

$g = f$!





$$\boxed{g = f}$$

<u>Lemma:</u> [Lemma 1.3.1 in WJ] [pg 11]

Given a step function

$$f = \sum_{i=1}^{n} k_i \cdot \chi_{J_i}$$

we can rewrite f as

$$f = \sum_{j=1}^{m} c_j \cdot \chi_{I_j}$$

where $I_s \cap I_t = \phi$ if $s \neq t$.

[That is, the I_j's are disjoint]

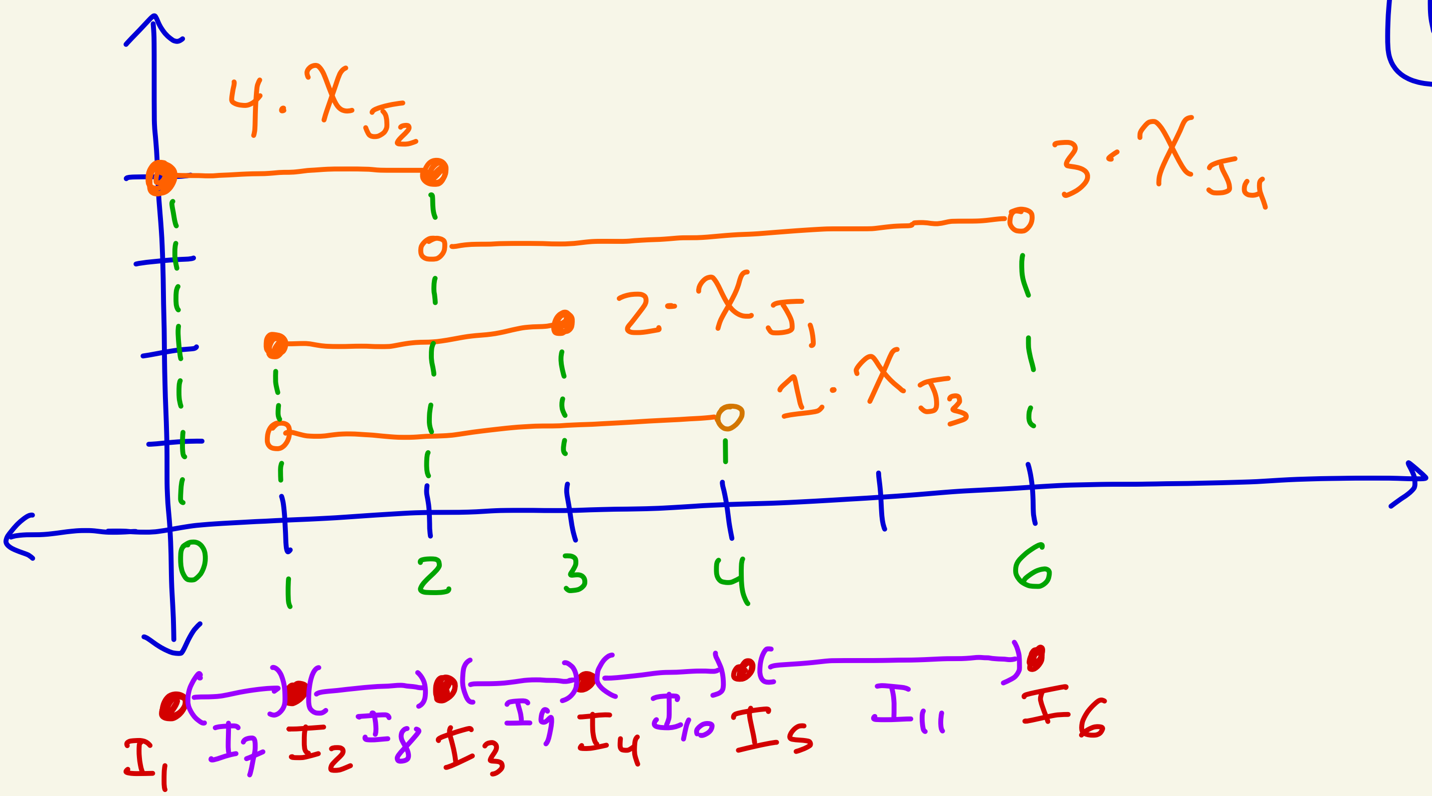
Ex of how proof will go

Let

$$f = 2\cdot\chi_{[1,3]} + 4\cdot\chi_{[0,2]} + 1\cdot\chi_{(1,4)} + 3\cdot\chi_{(2,6)}$$

$k_1 = 2,\ k_2 = 4,\ k_3 = 1,\ k_4 = 3$

$J_1 = [1,3],\ J_2 = [0,2],\ J_3 = (1,4),\ J_4 = (2,6)$



From the smallest to largest, list the endpoints of the J_i intervals.

$p_1 = 0$, $p_2 = 1$, $p_3 = 2$, $p_4 = 3$, $p_5 = 4$, $p_6 = 6$

Define:

$$\left.\begin{aligned} I_1 &= [0,0] = \{0\} \\ I_2 &= [1,1] = \{1\} \\ I_3 &= [2,2] = \{2\} \\ I_4 &= [3,3] = \{3\} \\ I_5 &= [4,4] = \{4\} \\ I_6 &= [6,6] = \{6\} \end{aligned}\right\} \text{endpoints} \qquad \left.\begin{aligned} I_7 &= (0,1) \\ I_8 &= (1,2) \\ I_9 &= (2,3) \\ I_{10} &= (3,4) \\ I_{11} &= (4,6) \end{aligned}\right\} \text{between endpoints}$$

Note all the I_j are disjoint

And,

$$f = \sum_{j=1}^{11} c_j \cdot \chi_{I_j}$$

where $c_j = \sum_{\substack{i \\ \text{where} \\ I_j \subseteq J_i}} k_i$

For example,

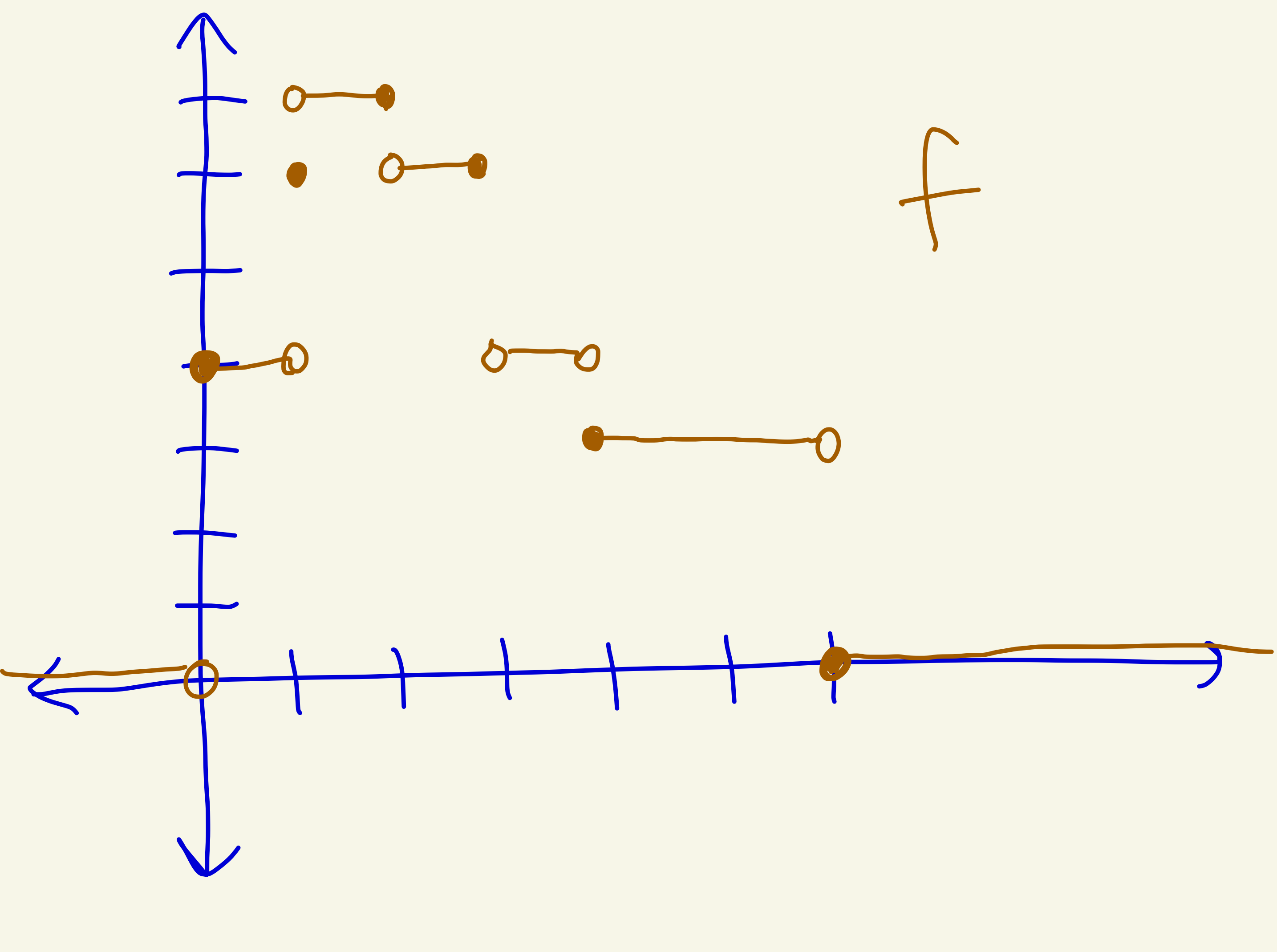
$c_1 = 4$ $\leftarrow$ $I_1 \subseteq J_2$, $k_2 = 4$

$c_2 = 2 + 4 = 6$ $\leftarrow$ $I_2 \subseteq J_1$, $k_1 = 2$; $I_2 \subseteq J_2$, $k_2 = 4$

$c_8 = 2 + 4 + 1 = 7$ $\leftarrow$ $I_8 \subseteq J_1$, $k_1 = 2$; $I_8 \subseteq J_2$, $k_2 = 4$; $I_8 \subseteq J_3$, $k_3 = 1$

We end up with

$$f = \boxed{4 \cdot \chi_{[0,0]}} + 6 \cdot \chi_{[1,1]} + \boxed{7 \cdot \chi_{[2,2]}} + \boxed{6 \cdot \chi_{[3,3]}} + \boxed{3 \cdot \chi_{[4,4]}} + 0 \cdot \chi_{[6,6]}$$
$$+ \boxed{4 \cdot \chi_{(0,1)}} + \boxed{7 \cdot \chi_{(1,2)}} + \boxed{6 \cdot \chi_{(2,3)}} + 4 \cdot \chi_{(3,4)} + \boxed{3 \cdot \chi_{(4,6)}}$$



We could simply this further by merging adjacent intervals with the same coefficient to get

$$f = 4 \cdot \chi_{[0,1)} + 6 \cdot \chi_{[1,1]} + 7 \cdot \chi_{(1,2]} + 6 \cdot \chi_{(2,3]} + 4 \cdot \chi_{(3,4)} + 3 \cdot \chi_{[4,6)}$$

This representation is unique in the sense that it cant be simplified further.