Math 5800

9/15/21

- Last class it was noted that HW 3 problem 4 is not the same as how we stated it in class. So I changed it to reflect the theorem from class. The proof is similar to before.

Lemma: Given a step function

$$f = \sum_{i=1}^{n} k_i \cdot \chi_{J_i}$$

we can re-write f as

$$f = \sum_{j=1}^{m} c_j \cdot \chi_{I_j}$$

where $I_s \cap I_t = \phi$ if $s \neq t$.

Proof: Let $p_1, p_2, \ldots, p_r$ be the r distinct endpoints of $J_1, J_2, \ldots, J_n$ where we arrange them so that $p_1 < p_2 < \ldots < p_r$.

Construct the following $m = 2r-1$ intervals

$I_1 = [p_1, p_1]$ $\quad I_{r+1} = (p_1, p_2)$

$I_2 = [p_2, p_2]$ $\quad I_{r+2} = (p_2, p_3)$

$\vdots$

$I_r = [p_r, p_r]$ $\quad I_{2r-1} = (p_{r-1}, p_r)$

Note by construction $I_s \cap I_t = \phi$ if $s \neq t$.

The I_j's partition $[p_1, p_r]$ into $2r-1$ disjoint sub-intervals.

For any of the new intervals I_j and original interval J_i, either $I_j \subseteq J_i$ or $I_j \cap J_i = \phi$

Let

$$c_j = \sum_{\substack{i \\ \text{where} \\ I_j \subseteq J_i}} k_i$$

If the sum is empty, then set $c_j = 0$.

[In our example from last time this would go with c_6]

Now we show $f = \sum_{j=1}^{m} c_j \chi_{I_j}$

This follow from the construction.

Let $p_1 \leq x \leq p_r$.

Then x is in exactly one of the I_s.

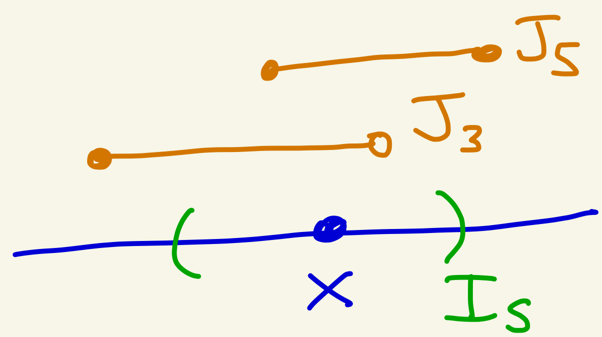
And by construction if x is in some J_i we have $I_s \subseteq J_i$

Hence,

$$f(x) = \sum_{i=1}^{n} k_i \cdot \chi_{J_i}(x)$$

$$= \sum_{\substack{i \\ \text{where} \\ x \in J_i}} k_i$$

$$= \sum_{\substack{i \\ \text{where} \\ I_s \subseteq J_i}} k_i = c_s.$$



$I_s \subseteq J_3$

$I_s \subseteq J_5$

Example pic

Thus, $f = \sum_j c_j \chi_{I_j}$ $\boxed{\diagup\diagup}$

<u>Note:</u> By merging adjacent interval terms with the same coefficients as we did in the example last time we can get a unique representation of f into the sum of the minimal number of disjoint terms.

<u>Theorem</u> [Thm 1.3.1 in WJ book]

Let f be a step function with two different representations

$$f = \sum_{j=1}^{m} c_j \chi_{I_j} = \sum_{i=1}^{n} k_i \chi_{J_i}$$

Then the integral of f is well-defined, that is

$$\int f = \sum_{j=1}^{m} c_j \, \ell(I_j) = \sum_{i=1}^{n} k_i \, \ell(J_i)$$

<u>proof:</u>

Let $q_1, q_2, \ldots, q_r$ be the r distinct endpoints of $I_1, I_2, \ldots, I_m, J_1, J_2, \ldots, J_n$

We arrange them in order so that

$$q_1 < q_2 < \cdots < q_r$$

Construct the following $2r-1$ intervals

$M_1 = [q_1, q_1]$

$M_2 = [q_2, q_2]$

$\vdots$

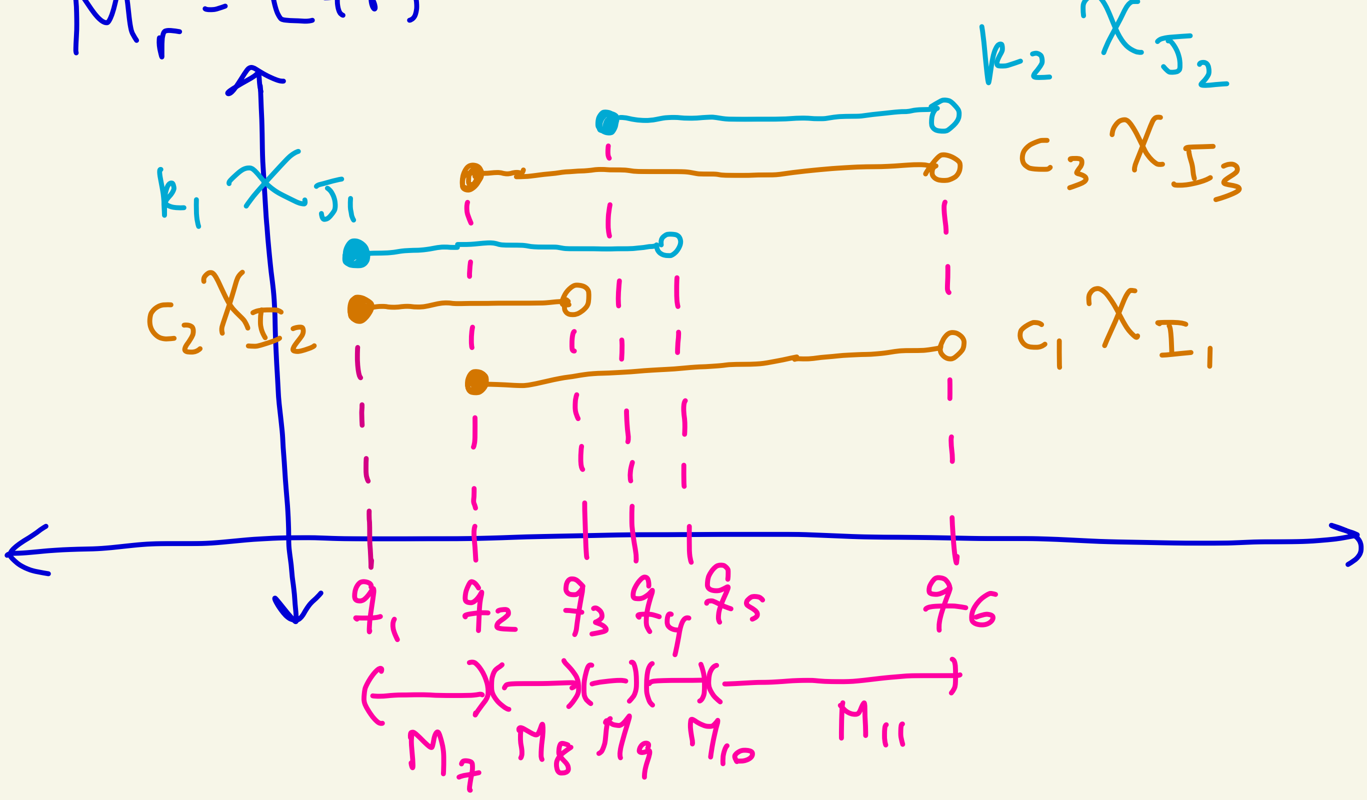
$M_r = [q_r, q_r]$

$M_{r+1} = (q_1, q_2)$

$M_{r+2} = (q_2, q_3)$

$\vdots$

$M_{2r-1} = (q_{r-1}, q_r)$



Note that $M_s \cap M_t = \phi$ if $s \neq t$

Given M_s and I_j either

$$M_s \subseteq I_j \text{ or } M_s \cap I_j = \phi$$

Given M_s and J_i either

$$M_s \subseteq J_i \text{ or } M_s \cap J_i = \phi.$$

Note that if $x \in M_s$ then

$$f(x) = \sum_{\substack{j \\ \text{where} \\ M_s \subseteq I_j}} c_j = \sum_{\substack{i \\ \text{where} \\ M_s \subseteq J_i}} k_i$$

Thus for each M_s define

$$\theta_s = \sum_{\substack{j \\ \text{where} \\ M_s \subseteq I_j}} c_j = \sum_{\substack{i \\ \text{where} \\ M_s \subseteq J_i}} k_i$$

Thus,

$$f = \sum_{s=1}^{2r-1} \theta_s \cdot \chi_{M_s}$$

This is a disjoint representation for f.

Claim: $\displaystyle\sum_{j=1}^{m} c_j \, \ell(I_j) = \sum_{s=1}^{2r-1} \theta_s \, \ell(M_s)$

pf of claim: By construction, for each j, we have $I_j = \bigcup\limits_{\substack{s \\ \text{where} \\ M_s \subseteq I_j}} M_s$

and the sum is disjoint.

And so,

$$\ell(I_j) = \sum_{\substack{s \\ \text{where} \\ M_s \subseteq I_j}} \ell(M_s)$$

Thus, $c_j \, \ell(I_j) = \sum_{\substack{s \\ \text{where} \\ M_s \subseteq I_j}} c_j \, \ell(M_s)$.

Summing over all the I_j's gives

$$\sum_{j=1}^{m} c_j \, \ell(I_j) = \underbrace{\sum_{j=1}^{m}}_{\text{sums over } I_j} \underbrace{\sum_{\substack{s \\ \text{where} \\ M_s \subseteq I_j}}}_{\text{sums over } M_s \text{ inside } I_j} c_j \, \ell(M_s)$$

$$= \underbrace{\sum_{s=1}^{2r-1}}_{\text{sums over } M_s} \underbrace{\sum_{\substack{j \\ \text{where} \\ M_s \subseteq I_j}}}_{\text{sums over } I_j \text{ containing } M_s} c_j \, \ell(M_s)$$

$$= \sum_{s=1}^{2r-1} \theta_s \, \ell(M_s) \qquad \boxed{\text{Claim}}$$

Claim: $\sum_{i=1}^{n} k_i \, \ell(J_i) = \sum_{s=1}^{2r-1} \theta_s \, \ell(M_s)$

pf: same as previous claim. $\boxed{\text{claim}}$

Combining the two claims,

$$\sum_{j=1}^{m} c_j \, \ell(I_j) = \sum_{s=1}^{2r-1} \theta_s \, \ell(M_s)$$

$$= \sum_{i=1}^{n} k_i \, \ell(J_i).$$

∎