Math 5800

9/22/21

**Theorem:** [Weir - Thm 1 - pg 31]

Let $(\varphi_n)_{n=1}^{\infty}$ be a non-decreasing sequence of step functions. Suppose also that the sequence

$$\underbrace{\left(\int \varphi_n\right)_{n=1}^{\infty}}_{\boxed{\int\varphi_1,\ \int\varphi_2,\ \int\varphi_3,\ \int\varphi_4,\ \ldots \text{ sequence of real numbers}}} \text{ converges.}$$

Then

$$S = \left\{ x \in \mathbb{R} \;\middle|\; \underbrace{(\varphi_n(x))_{n=1}^{\infty}}_{\boxed{\varphi_1(x),\ \varphi_2(x),\ \varphi_3(x),\ \ldots}} \text{ does not converge} \right\}$$

is a set of measure zero.

Or equivalently

$$\mathbb{R} - S = \left\{ x \;\middle|\; (\varphi_n(x))_{n=1}^{\infty} \text{ converges} \right\}$$

is an almost everywhere set

Proof:

Claim 1: We may assume that $\varphi_n(x) \geq 0$ for all $n \geq 1$ and $x \in \mathbb{R}$

pf of claim 1:

Consider the sequence of step functions

$$(\varphi_n - \varphi_1)_{n=1}^{\infty}.$$

That is,

$$\varphi_1 - \varphi_1,\ \varphi_2 - \varphi_1,\ \varphi_3 - \varphi_1,\ \varphi_4 - \varphi_1,\ \ldots$$

Since $(\varphi_n)_{n=1}^{\infty}$ is non-decreasing we know that $\varphi_n(x) \geq \varphi_1(x)$ for all $n \geq 1$ and $x \in \mathbb{R}$.

Thus, $\varphi_n(x) - \varphi_1(x) \geq 0$.

So, $\boxed{(\varphi_n - \varphi_1)(x) \geq 0.}$

Also,

$$
\begin{aligned}
(\varphi_{n+1} - \varphi_1)(x) &= \varphi_{n+1}(x) - \varphi_1(x) \\
&\geq \varphi_n(x) - \varphi_1(x) \\
&= (\varphi_n - \varphi_1)(x)
\end{aligned}
$$

$\boxed{\varphi_{n+1} \geq \varphi_n}$

Thus, $(\varphi_n - \varphi_1)_{n=1}^{\infty}$ is a non-decreasing sequence.

Since $\left(\int \varphi_n\right)_{n=1}^{\infty}$ converges and

$$\int (\varphi_n - \varphi_1) = \int \varphi_n - \underbrace{\int \varphi_1}_{\text{constant}}$$

we know that

$$\left(\int (\varphi_n - \varphi_1)\right)_{n=1}^{\infty} \text{ converges.}$$

And,

$$(\varphi_n - \varphi_1)(x) = \varphi_n(x) - \overbrace{\varphi_1(x)}^{\text{constant}}$$

converges as $n \to \infty$ iff $\varphi_n(x)$ converges.

Thus,

$$T = \{ x \mid (\varphi_n - \varphi_1)(x) \text{ does not converge} \}$$

equals $S$.

So, $T$ has measure zero iff $S$ does.

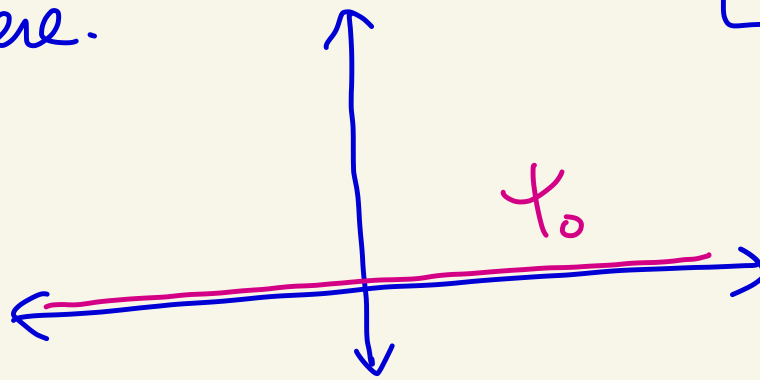
Thus, we could prove the theorem by replacing $(\varphi_n)_{n=1}^{\infty}$ by $(\varphi_n - \varphi_1)_{n=1}^{\infty}$.

But we won't do this.

We will just assume $\varphi_n(x) \geq 0$ for all $n \geq 1$ and $x \in \mathbb{R}$

$\boxed{\text{Claim 1}}$

Let $\psi_0$ be the step function that is zero everywhere.




Then, $\underbrace{\psi_0(x)}_{0} \leq \varphi_n(x) \leq \varphi_{n+1}(x)$ for all $x$ and $n$.

$\boxed{\text{claim 1}}$

Thus, $\underbrace{\int \psi_0(x)}_{0} \leq \int \varphi_n \leq \int \varphi_{n+1}$ for all $n$.

So, $\left(\int \varphi_n\right)_{n=1}^{\infty}$ is a non-decreasing, non-negative, convergent sequence of real numbers.

Since $\left(\int \varphi_n\right)_{n=1}^{\infty}$ converges, it is bounded.

So, $\exists K > 0$ where $\boxed{0 \leq \int \varphi_n \leq K}$ for all $n \geq 1$.

Let $\varepsilon > 0$.

Define

$$S_n^{\varepsilon} = \left\{ x \in \mathbb{R} \;\middle|\; \varphi_n(x) \geq \frac{K}{\varepsilon} \right\}.$$

Claim 2: $S_n^{\varepsilon} \in \mathcal{R}$ for $n \geq 1$

proof of claim 2:

Let $\varphi_n = c_1 \chi_{I_1} + c_2 \chi_{I_2} + \ldots + c_r \chi_{I_r}$

where $I_1, I_2, \ldots, I_r$ are disjoint bounded intervals.

Pick the indices

$$1 \leq i_1 < i_2 < \ldots < i_t \leq r$$

where

$$c_{i_1}, c_{i_2}, \ldots, c_{i_t} \geq \frac{K}{\varepsilon}.$$

So, $\varphi_n(x) \geq \frac{K}{\varepsilon}$ when $x \in I_{i_s}$, $1 \leq s \leq t$.

Thus,

$$S_n^\varepsilon = I_{i_1} \cup I_{i_2} \cup \dots \cup I_{i_t}$$

So, $S_n^\varepsilon \in \mathcal{R}$.

Note it could be that $\varphi_n(x) \geq \frac{K}{\varepsilon}$ is never satisfied.

In that special case,

$$S_n^\varepsilon = \phi = (1,1) \in \mathcal{R}.$$

$\boxed{\text{claim 2}}$

<u>Claim 3:</u> $\left(\frac{K}{\varepsilon}\right) \chi_{S_n^\varepsilon}(x) \leq \varphi_n(x)$

for all $x \in \mathbb{R}$ and $n \geq 1$.

<u>pf of claim 3:</u>

If $x \in S_n^\varepsilon$, then $\left(\frac{K}{\varepsilon}\right) \overbrace{\chi_{S_n^\varepsilon}(x)}^{1} = \left(\frac{K}{\varepsilon}\right) \overset{x \in S_n^\varepsilon}{\leq} \varphi_n(x)$

If $x \notin S_n^\varepsilon$, then $\left(\frac{K}{\varepsilon}\right) \underbrace{\chi_{S_n^\varepsilon}(x)}_{0} = 0 \underset{\text{claim 1}}{\leq} \varphi_n(x)$

$\boxed{\text{Claim 3}}$

Therefore,

$$\left(\frac{k}{\varepsilon}\right) \ell(S_n^\varepsilon) = \left(\frac{k}{\varepsilon}\right) \int \chi_{S_n^\varepsilon} = \int \left(\frac{k}{\varepsilon}\right) \chi_{S_n^\varepsilon}$$

$S_n^\varepsilon \in \mathcal{R}$

$\ell(S_n^\varepsilon) = \int \chi_{S_n^\varepsilon}$

$$\leq \int \varphi_n$$

claim 3

Thus, $\left(\frac{k}{\varepsilon}\right) \ell(S_n^\varepsilon) \leq \int \varphi_n \leq K.$

So, $\boxed{\ell(S_n^\varepsilon) \leq \varepsilon}$ for all $n \geq 1$.

Claim 4: $S_n^\varepsilon \subseteq S_{n+1}^\varepsilon$ for all $n \geq 1$. pg 9

pf of claim:

Let $x \in S_n^\varepsilon$.

$(\varphi_n)_{n=1}^\infty$ non-decreasing

Then, $\frac{K}{\varepsilon} \leq \varphi_n(x) \leq \varphi_{n+1}(x)$

Thus, $x \in S_{n+1}^\varepsilon$ $\boxed{\text{Claim 4}}$

Thus,

$$S_1^\varepsilon \subseteq S_2^\varepsilon \subseteq S_3^\varepsilon \subseteq S_4^\varepsilon \subseteq \cdots$$

Let

$$S^\varepsilon = \bigcup_{n=1}^\infty S_n^\varepsilon$$

**Claim 5:** $S \subseteq S^{\varepsilon}$

Recall $S = \{x \in \mathbb{R} \mid (\varphi_n(x))_{n=1}^{\infty} \text{ diverges}\}$

**pf of claim:**

Let $x \in S$.

Then, $(\varphi_n(x))_{n=1}^{\infty}$ does not converge.

Because $(\varphi_n(x))_{n=1}^{\infty}$ is non-decreasing by the monotone convergence theorem $(\varphi_n(x))_{n=1}^{\infty}$ is not bounded.

Thus, there must exist some $N > 0$ where $\varphi_N(x) \geq \frac{K}{\varepsilon}$.

Hence, $x \in S_N^{\varepsilon} \subseteq \underbrace{S^{\varepsilon}}_{S^{\varepsilon} = \bigcup_{n=1}^{\infty} S_n^{\varepsilon}}$.

$\boxed{\text{claim 5}}$

Therefore, we can show that $S^{\varepsilon}$ has measure zero and then this will imply that $S$ has measure zero.

By a theorem from last time, since $S_{n+1}^{\varepsilon} \in \mathcal{R}$ and $S_n^{\varepsilon} \in \mathcal{R}$ for all $n \geq 1$ we know that

$$S_{n+1}^{\varepsilon} - S_n^{\varepsilon} \in \mathcal{R}.$$

Because

$$S_1^{\varepsilon} \subseteq S_2^{\varepsilon} \subseteq S_3^{\varepsilon} \subseteq S_4^{\varepsilon} \subseteq \cdots$$

and $S^{\varepsilon} = \bigcup_{n=1}^{\infty} S_n^{\varepsilon}$ we know that

$$S^{\varepsilon} = S_1^{\varepsilon} \cup (S_2^{\varepsilon} - S_1^{\varepsilon}) \cup (S_3^{\varepsilon} - S_2^{\varepsilon}) \cup \cdots$$

is a disjoint union giving $S$.

From above we can write

$$\left.\begin{aligned} S_1^{\varepsilon} &= I_1 \cup I_2 \cup \cdots \cup I_{n_1} \\ S_2^{\varepsilon} - S_1^{\varepsilon} &= I_{n_1+1} \cup I_{n_1+2} \cup \cdots \cup I_{n_2} \\ S_3^{\varepsilon} - S_2^{\varepsilon} &= I_{n_2+1} \cup I_{n_2+2} \cup \cdots \cup I_{n_3} \\ &\vdots \end{aligned}\right\} \text{these are all in } \mathbb{R}$$

where all the above $I_k$ are bounded intervals and disjoint from each other.

Then,

$$S^{\varepsilon} = S_1^{\varepsilon} \cup (S_2^{\varepsilon} - S_1^{\varepsilon}) \cup (S_3^{\varepsilon} - S_2^{\varepsilon}) \cup \cdots$$

$$= \bigcup_{k=1}^{\infty} I_k$$

So the $I_k$ cover $S^{\varepsilon}$.

Given $n \geq 1$, there exists $j$ where $n \leq n_j$ and so for this $n$ we have that



$$\sum_{k=1}^{n} \ell(I_k) \leq \sum_{k=1}^{n_j} \ell(I_k) = \ell(S_j^{\varepsilon}) \leq \varepsilon$$

$$S_j^{\varepsilon} = (S_j^{\varepsilon} - S_{j-1}^{\varepsilon}) \cup (S_{j-1}^{\varepsilon} - S_{j-2}^{\varepsilon}) \cup \cdots \cup (S_2^{\varepsilon} - S_1^{\varepsilon}) \cup S_1^{\varepsilon}$$

Thus, $\displaystyle\sum_{k=1}^{n} \ell(I_k) \leq \varepsilon$ for all $n \geq 1$.

So, $\displaystyle\sum_{k=1}^{\infty} \ell(I_k) \leq \varepsilon$.

Thus, $S^{\varepsilon}$ has measure zero.

So, $S$ has measure zero. $\blacksquare$