Math 5800

9/27/21

$\boxed{\text{Test 1 is on Monday Oct 18 Test 1 covers HW 3 and HW 4}}$

No class on Test day.
Test is done on canvas.
Test will appear at 5am on Monday 10/18 and dissapear at 12pm noon on Tuesday 10/19. During that time period you pick a 2.5 hour time window to take the test, scan, and upload your answers [2 hrs for test, 30 min to scan]. Canvas will time you once you open the test.

I put a "practice taking a test" Module in case you haven't taken a test on canvas before to see what its like to download an exam and upload your solutions.

Try it out if needed.

In the theorem from last time we could have assumed that $\left(\int \varphi_n\right)_{n=1}^{\infty}$ was bounded.

That is:

<u>Theorem:</u> Let $(\varphi_n)_{n=1}^{\infty}$ be a non-decreasing sequence of step functions.

Then, $\left(\int \varphi_n\right)_{n=1}^{\infty}$ converges iff $\left(\int \varphi_n\right)_{n=1}^{\infty}$ is bounded.

<u>proof:</u>

($\Rightarrow$) If $\left(\int \varphi_n\right)_{n=1}^{\infty}$ converges, then by 4650 HW, $\left(\int \varphi_n\right)_{n=1}^{\infty}$ is bounded.

($\Leftarrow$) Suppose $\left(\int \varphi_n\right)_{n=1}^{\infty}$ is bounded.

Since $(\varphi_n)_{n=1}^{\infty}$ is non-decreasing we know that $\varphi_n(x) \leq \varphi_{n+1}(x)$ for all $n \geq 1$ and $x \in \mathbb{R}$.

By a theorem in class,

$$\int \varphi_n \leq \int \varphi_{n+1} \quad \text{for all } n \geq 1.$$

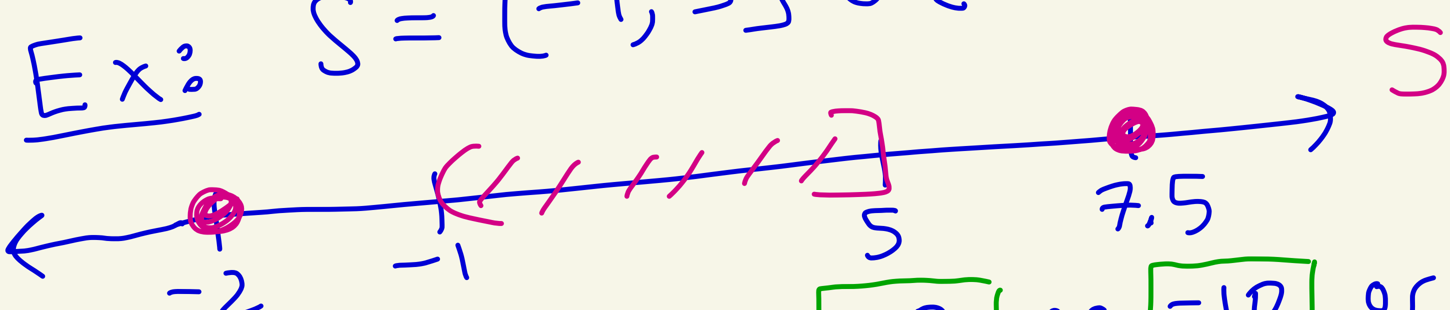
Therefore, $\left(\int \varphi_n\right)_{n=1}^{\infty}$ is a non-decreasing bounded sequence of real numbers.

By the monotone convergence theorem from 4650, $\left(\int \varphi_n\right)_{n=1}^{\infty}$ converges. $\blacksquare$

Topic 5 - More 4650 Review

Def: Let $S \subseteq \mathbb{R}$ with $S \neq \emptyset$.
Let $M \in \mathbb{R}$.
We say that M is an **upper bound** for S if $x \leq M$ for all $x \in S$.
We say that M is a **lower bound** for S if $M \leq x$ for all $x \in S$.

Ex: $S = (-1, 5] \cup \{-2, 7.5\}$



some lower bounds: $\boxed{-2}$ or $\boxed{-10}$ or ...
some upper bounds: $\boxed{7.5}$ or $\boxed{10{,}000{,}000}$ or ...

Def: Let $S \subseteq \mathbb{R}$ and $S \neq \emptyset$.
Let $M \in \mathbb{R}$.
We say that M is the least upper bound, or supremum, of S if

① M is an upper bound for S

and ② for any upper bound B of S, we have $M \leq B$

M is the smallest upper bound

If such an M exists, we write $M = \sup(S)$

Def: Let $S \subseteq \mathbb{R}$ and $S \neq \emptyset$.
Let $M \in \mathbb{R}$.
We say that M is the greatest lower bound, or infimum, of S if

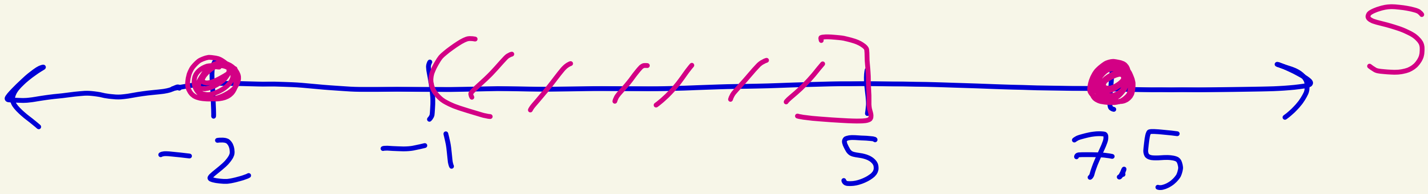
① M is a lower bound for S

and ② for any lower bound B of S, we have $B \leq M$

M is the biggest lower bound

If such an M exists, we write
$$M = \inf(S).$$

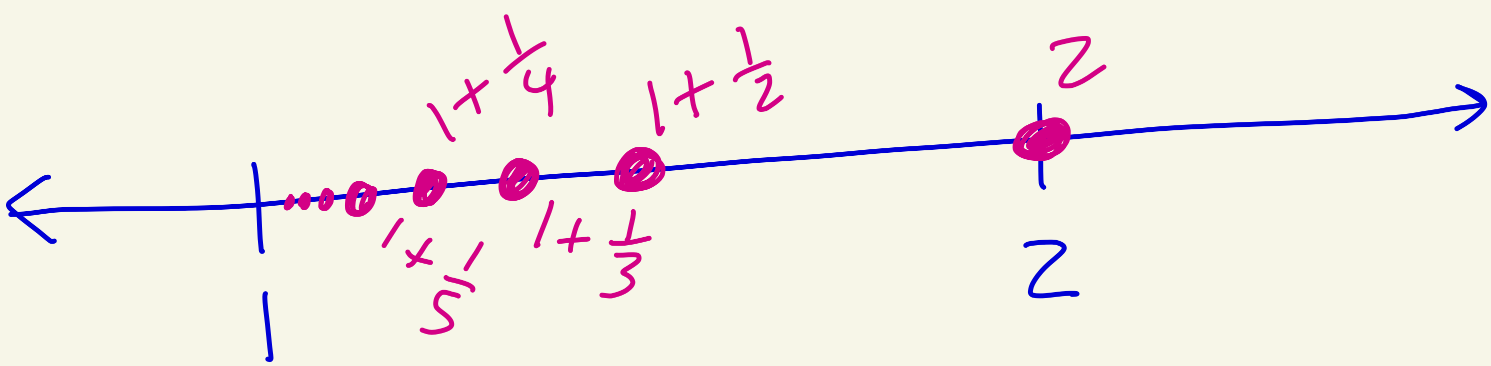
Ex: $S = (-1, 5] \cup \{-2, 7.5\}$



$\inf(S) = -2$

$\sup(S) = 7.5$

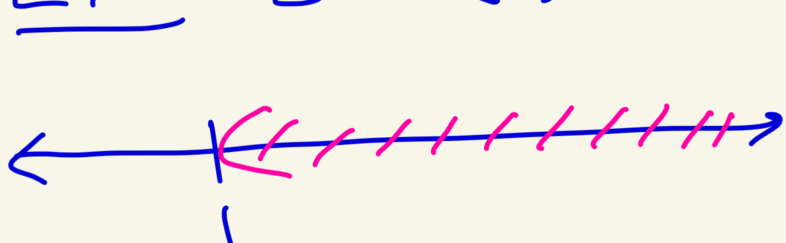
Ex: $S = \left\{ 1 + \frac{1}{n} \;\middle|\; n = 1, 2, 3, 4, \ldots \right\}$



$\inf(S) = 1$

$\sup(S) = 2$

Ex: $S = (1, \infty)$



$\inf(S) = 1$ and $\sup(S)$ does not exist

Completness axiom for $\mathbb{R}$

Let $S \subseteq \mathbb{R}$ with $S \neq \emptyset$.

① If S is bounded from above [that is, an upper bound for S exists], then $\sup(S)$ exists.

② If S is bounded from below [that is, a lower bound for S exists], then $\inf(S)$ exists.

proof: You would construct $\mathbb{R}$ via Dedekind cuts of $\mathbb{Q}$ or via cauchy sequences of $\mathbb{Q}$. Then you prove this axiom is true. $\blacksquare$

<u>Theorem:</u> Let $S \subseteq \mathbb{R}$ with $S \neq \emptyset$.

① If S has an infimum, then the infimum is unique.

② If S has a supremum, then the supremum is unique.

<u>pf:</u> 4650 HW $\blacksquare$

Theorem: Let $A, B \subseteq \mathbb{R}$ with $A \neq \emptyset$ and $B \neq \emptyset$.
Suppose $A \subseteq B$.

1. If $\inf(B)$ exists, then $\inf(A)$ exists and $\inf(B) \leq \inf(A)$.
2. If $\sup(B)$ exists, then $\sup(A)$ exists and $\sup(A) \leq \sup(B)$.

proof:

1. Suppose $\inf(B)$ exists.
Then $\inf(B) \leq b$ for all $b \in B$.
Since $A \subseteq B$, this means also that $\inf(B) \leq a$ for all $a \in A$.
By the completeness axiom since A is bounded below, $\inf(A)$ exists.

We saw that $\inf(B)$ is a lower bound for A.

Since $\inf(A)$ is the greatest lower bound for A we know that $\inf(B) \leq \inf(A)$

[prop 2 of infimum]

(2) Similar to part 1.

$\blacksquare$

Topic 6 – Sequences of functions and the standard construction

<u>Def:</u> Let $D \subseteq \mathbb{R}$.
Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions where $f_n : D \to \mathbb{R}$ for $n \geq 1$. Let $f : D \to \mathbb{R}$.
We say that $(f_n)_{n=1}^{\infty}$ converges to f <u>pointwise on D</u> if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for any $x \in D$.
If this is the case we write "$\lim f_n = f$ pointwise on D"
or "$f_n \to f$ pointwise on D"

So, $f_n \to f$ pointwise on D means that if $x \in D$ is fixed then

$f_1(x), f_2(x), f_3(x), \ldots$

converges to $f(x)$.

Ex: Let $f_n(x) = \frac{x}{n}$ for $n \geq 1$. Let $f(x) = 0$ for all $x \in \mathbb{R}$.

Claim: $f_n \to f$ pointwise for all $x \in \mathbb{R}$.

pf of claim: Let $x \in \mathbb{R}$.

Then,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = x \cdot \lim_{n\to\infty} \frac{1}{n}$$

$$= x \cdot 0 = 0 = f(x).$$

$\blacksquare$