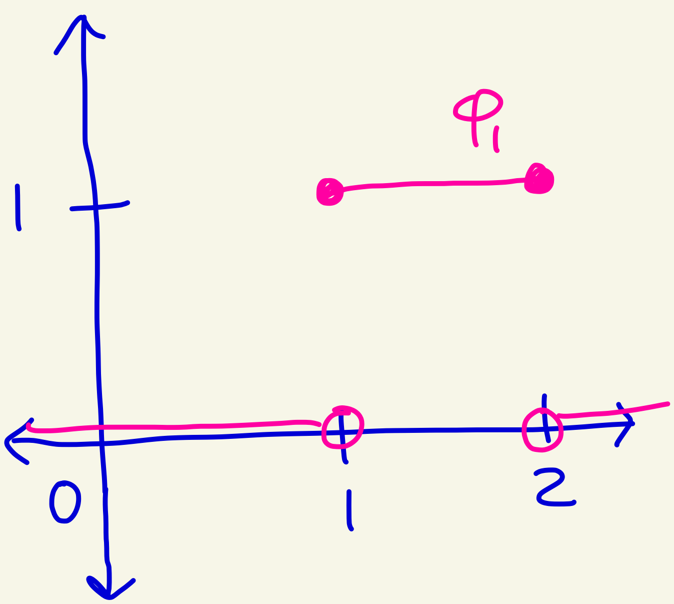
Math 5800

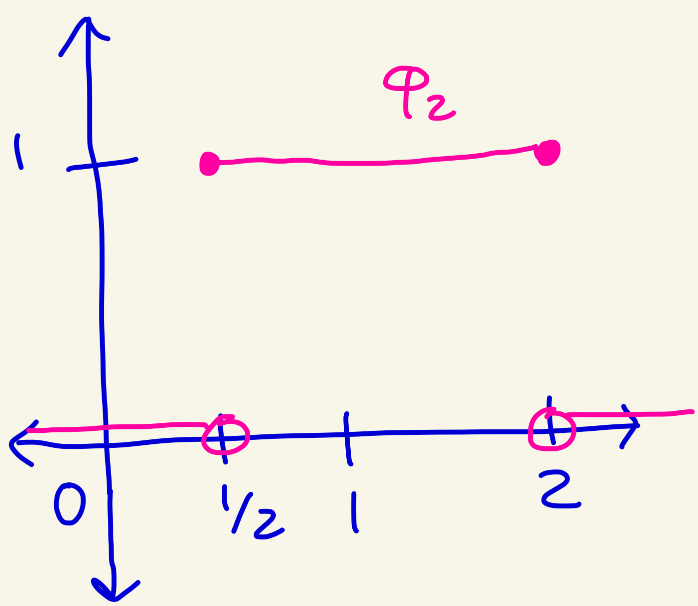
9/29/21

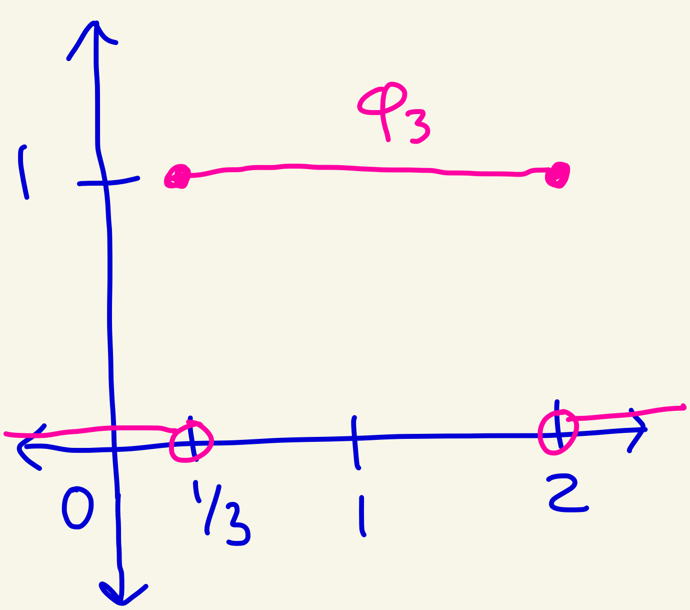
We talked about Test 1 on Monday.

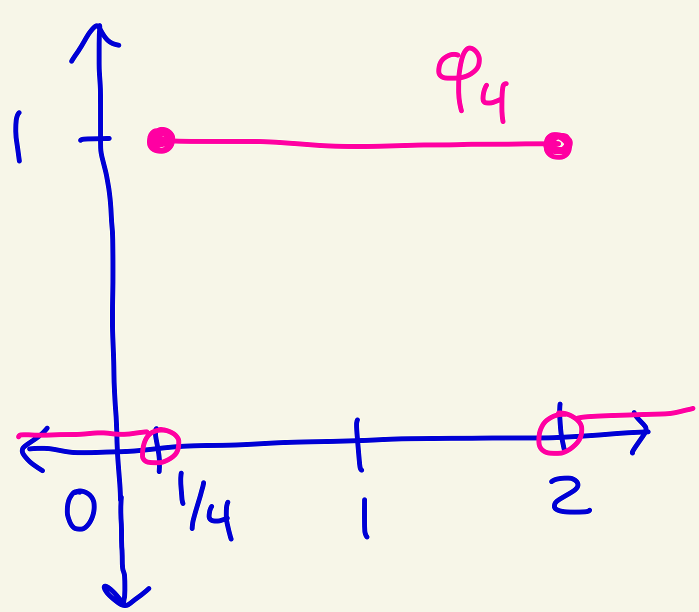
Ex: Let $\varphi_n = \chi_{[\frac{1}{n}, 2]}$

We showed previously that $(\varphi_n)_{n=1}^{\infty}$ was non-decreasing.



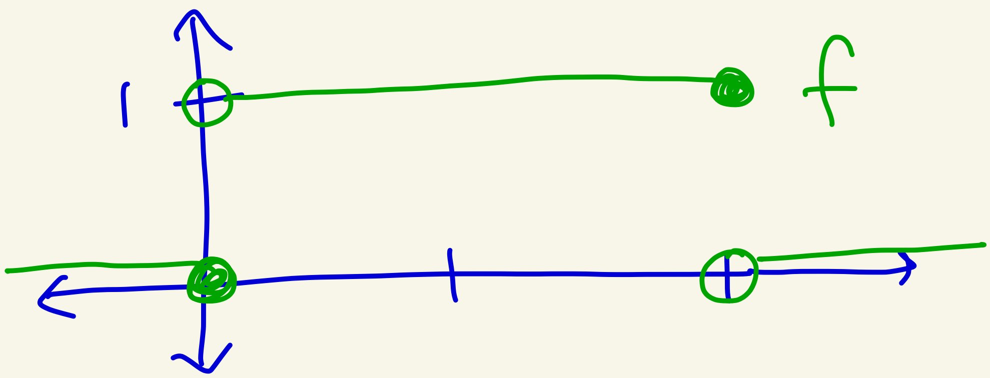






Let

$$f(x) = \chi_{(0,2]}(x) = \begin{cases} 1 & \text{if } x \in (0,2] \\ 0 & \text{otherwise} \end{cases}$$



$\boxed{\text{Claim: } \varphi_n \to f \text{ pointwise on all of } \mathbb{R}}$

<u>proof:</u> Let $x \in \mathbb{R}$.

<u>case 1:</u> Suppose $x \notin (0,2]$.

So, $x \le 0$ or $x > 2$.

Then, $\varphi_n(x) = \chi_{[\frac{1}{n},2]}(x) = 0$

for all $n \geq 2$.

Thus, $\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} 0 = 0 = f(x)$

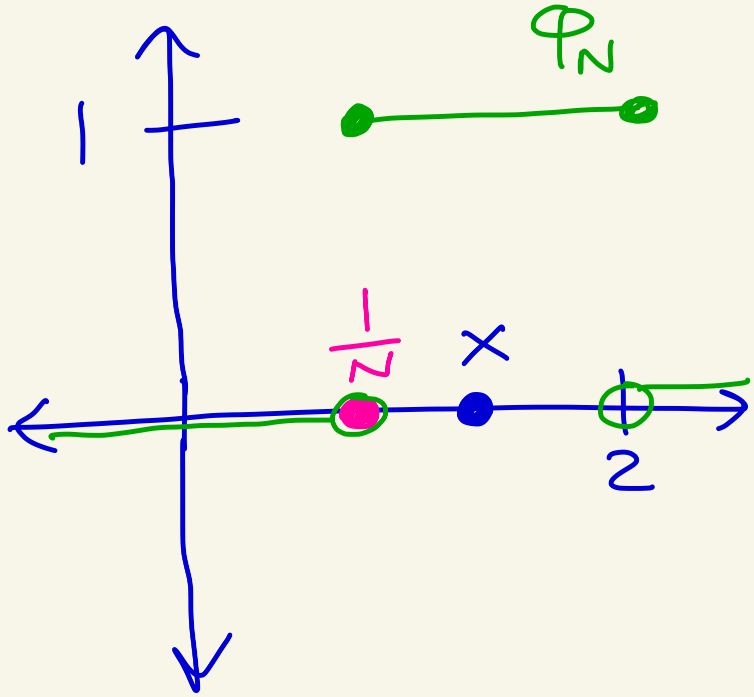
case 2: Suppose $x \in (0,2]$

Then $0 < x \leq 2$. Let $\varepsilon > 0$.

Pick $N > 0$ where $0 < \frac{1}{N} \leq x$.

Then, $\varphi_N(x) = 1$.

If $n \geq N$ then



$$\varphi_n(x) = \chi_{[\frac{1}{n},2]}(x)$$

$$= \chi_{[\frac{1}{N},2]}(x)$$

$$= 1$$

because $[\frac{1}{N},2] \subseteq [\frac{1}{n},2]$.

Thus, if $n \geq N$ then

$$|\varphi_n(x) - f(x)| = |1-1|$$

$$= 0 < \varepsilon.$$

So, $\lim_{n\to\infty} \varphi_n(x) = f(x)$. $\boxed{\diagup\diagup}$

Def: Let $S \subseteq \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$. We say that f is <u>bounded</u> on S if there exists $M > 0$ where

$$\underbrace{|f(x)| \leq M}_{-M \leq f(x) \leq M}$$

for all $x \in S$.

From WJ book

Def: (Standard construction for bounded functions on $[a,b]$)

Let $a, b \in \mathbb{R}$ with $a < b$.
Let $f : \mathbb{R} \to \mathbb{R}$ where f is bounded on $[a,b]$.

Given $n \geq 1$ we will define a step function γ_n.

Divide $[a,b]$ into 2^n subintervals each of with $\Delta_n = \frac{b-a}{2^n}$

as follows:

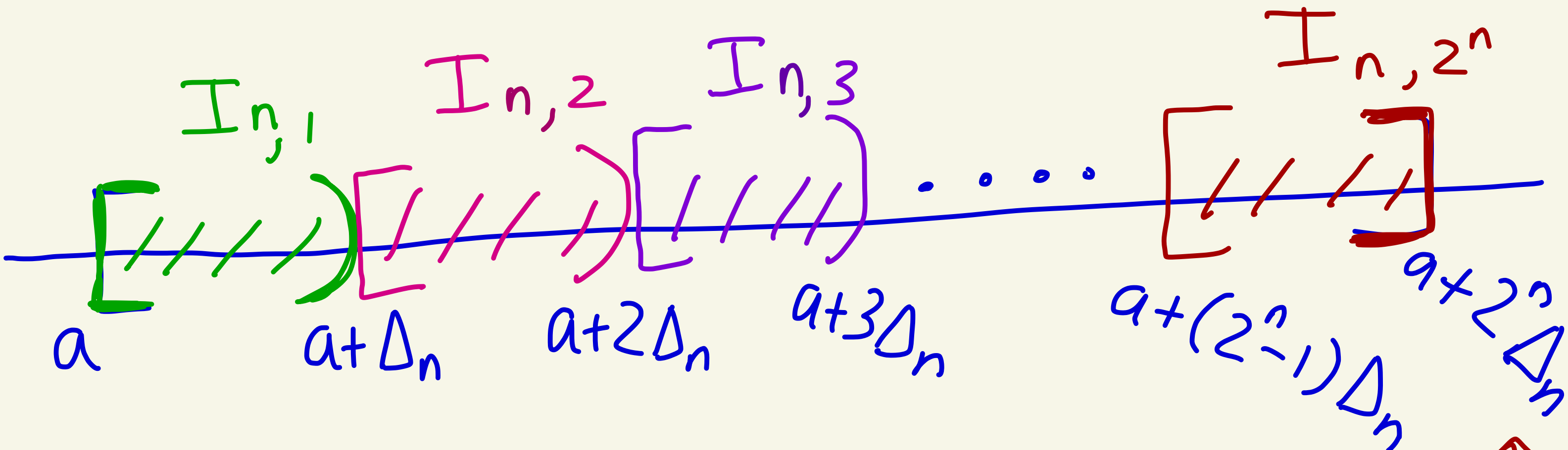
Let

$$I_{n,k} = [a+(k-1)\Delta_n \,,\, a+k\Delta_n)$$

where $k = 1, 2, 3, \ldots, 2^n - 1$,

and

$$I_{n,2^n} = [a+(2^n-1)\Delta_n \,,\, a+2^n\Delta_n]$$



$$\boxed{\text{Notice } a+2^n\Delta_n = a+2^n\frac{(b-a)}{2^n} = b}$$

Now define

$$\gamma_n = \sum_{k=1}^{2^n} m_{n,k} \cdot I_{n,k}$$

where $m_{n,k} = \inf\{ f(t) \mid t \in I_{n,k} \}$

Note $m_{n,k}$ exists since f is bounded on each $I_{n,k}$ by assumption

The sequence $(\gamma_n)_{n=1}^{\infty}$ is called the standard construction for f on $[a,b]$

Ex: Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=x$.
Let $[a,b]=[0,1]$.
Let's construct the standard construction for f on $[0,1]$.

$\boxed{n=1}$ $\Delta_n=\frac{b-a}{2^n}=\frac{1}{2}$

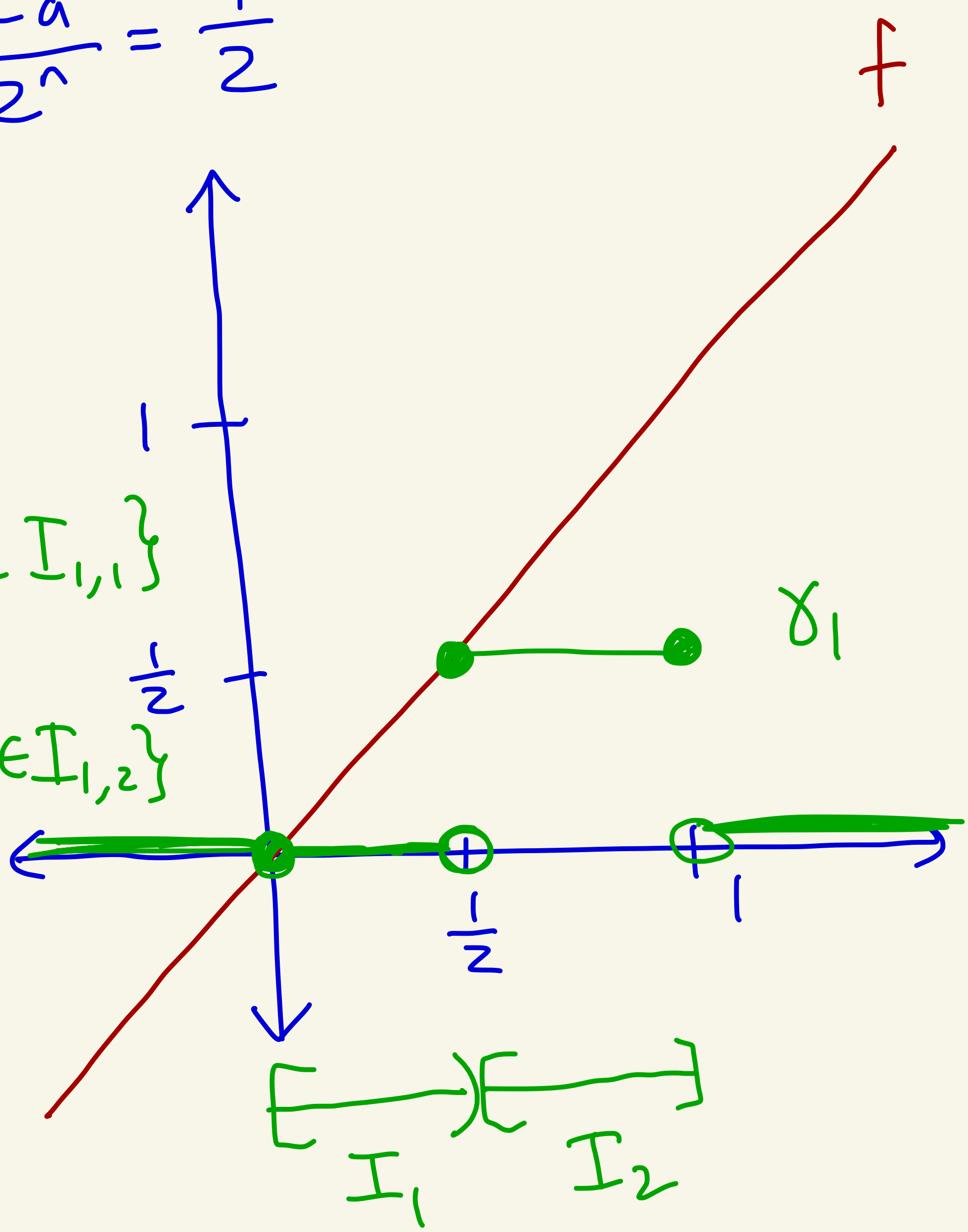
$I_{1,1}=[0,\frac{1}{2})$

$I_{1,2}=[\frac{1}{2},1]$

$m_{1,1}=\inf\{f(t)\mid t\in I_{1,1}\}$
$=0$

$m_{1,2}=\inf\{f(t)\mid t\in I_{1,2}\}$
$=1/2$

$\gamma_1=0\cdot\chi_{I_{1,1}}$
$+\frac{1}{2}\cdot\chi_{I_{1,2}}$



$\boxed{n=2}$ $\Delta_2 = \frac{b-a}{2^n} = \frac{1-0}{2^2} = \frac{1}{4}$

$I_{2,1} = [0, \frac{1}{4})$

$I_{2,2} = [\frac{1}{4}, \frac{1}{2})$

$I_{2,3} = [\frac{1}{2}, \frac{3}{4})$

$I_{2,4} = [\frac{3}{4}, 1]$

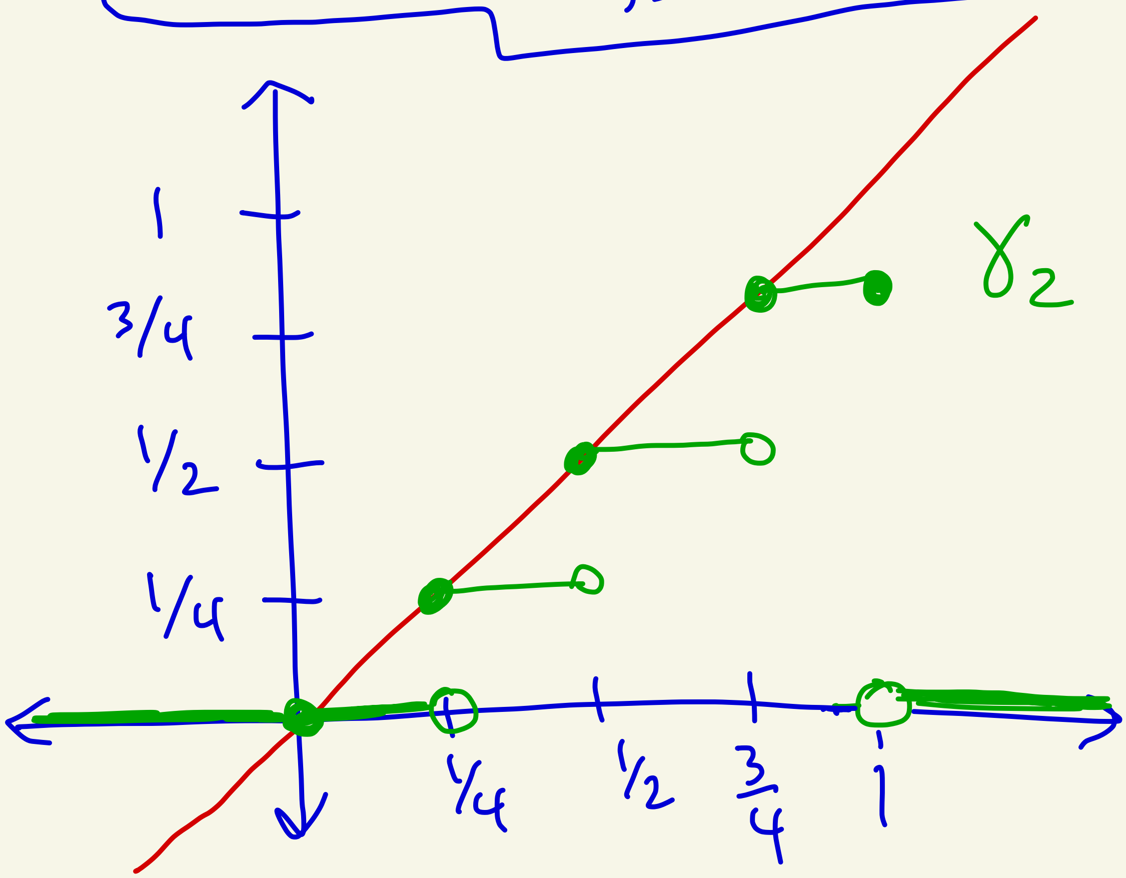
$m_{2,1} = \inf\{f(t) \mid t \in I_{2,1}\} = 0$

$m_{2,2} = \frac{1}{4}$

$m_{2,3} = 1/2$

$m_{2,4} = 3/4$

$$\boxed{\gamma_2 = 0 \cdot \chi_{I_{2,1}} + \frac{1}{4} \cdot \chi_{I_{2,2}} + \frac{1}{2} \cdot \chi_{I_{2,3}} + \frac{3}{4} \cdot \chi_{I_{2,4}}}$$

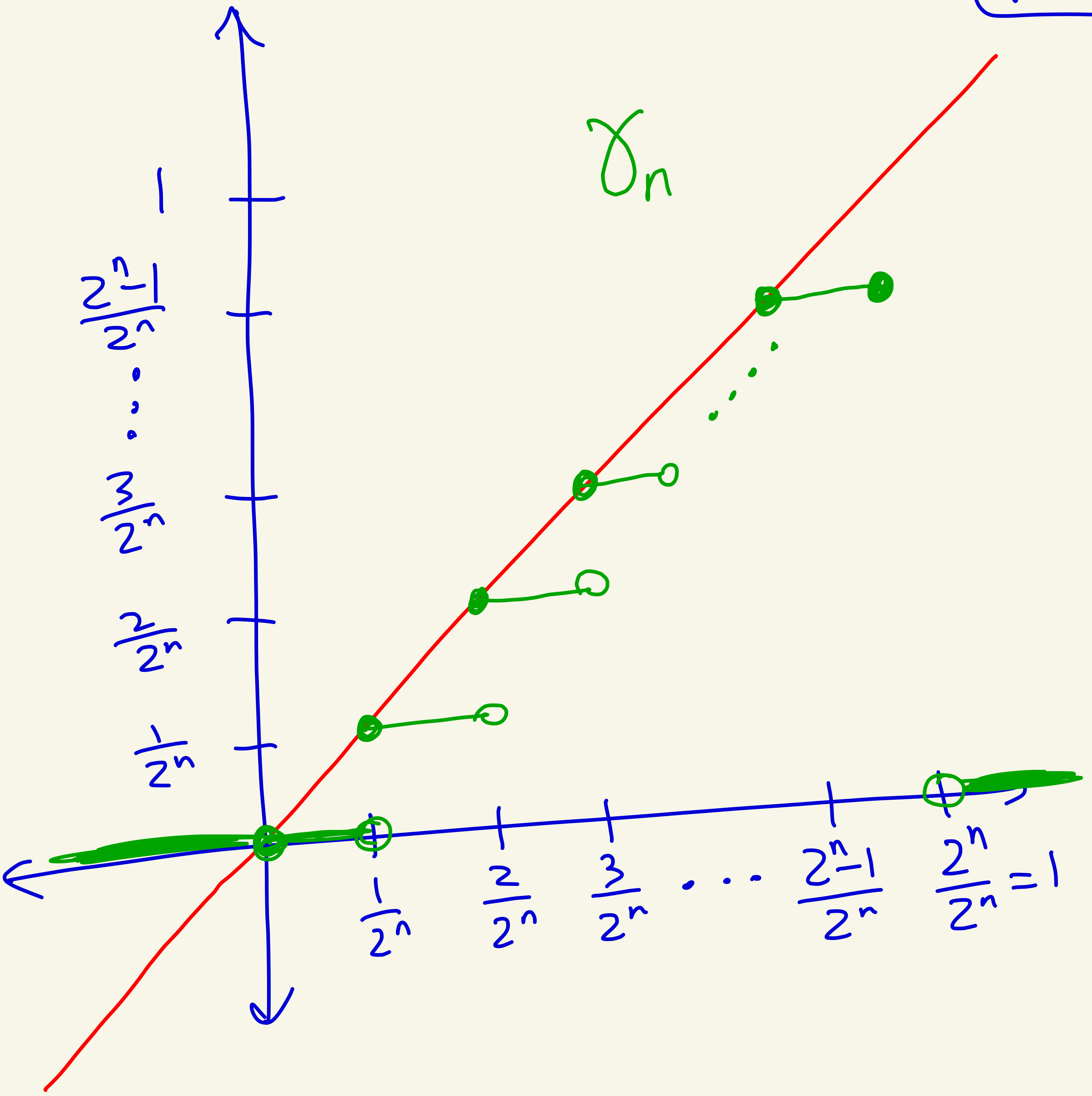


For general n we have that $\Delta_n = \frac{1-0}{2^n} = \frac{1}{2^n}$

$$\left.\begin{array}{ll} I_{n,1} = \left[0, \frac{1}{2^n}\right) & m_{n,1} = 0 \\ I_{n,2} = \left[\frac{1}{2^n}, \frac{2}{2^n}\right) & m_{n,2} = \frac{1}{2^n} \\ I_{n,3} = \left[\frac{2}{2^n}, \frac{3}{2^n}\right) & m_{n,3} = \frac{2}{2^n} \\ \vdots & \vdots \\ I_{n,2^n} = \left[\frac{2^n-1}{2^n}, 1\right] & m_{n,2^n} = \frac{2^n-1}{2^n} \end{array}\right\}$$

left end-points since f is an increasing function

$$\gamma_n = 0 \cdot \chi_{I_{n,1}} + \frac{1}{2^n} \cdot \chi_{I_{n,2}} + \frac{2}{2^n} \chi_{I_{n,3}} + \dots + \frac{2^n-1}{2^n} \cdot \chi_{I_{n,2^n}}$$


γ_n
1
$\frac{2^n-1}{2^n}$
$\frac{3}{2^n}$
$\frac{2}{2^n}$
$\frac{1}{2^n}$
$\frac{1}{2^n}$
$\frac{2}{2^n}$
$\frac{3}{2^n}$
$\frac{2^n-1}{2^n}$
$\frac{2^n}{2^n}=1$