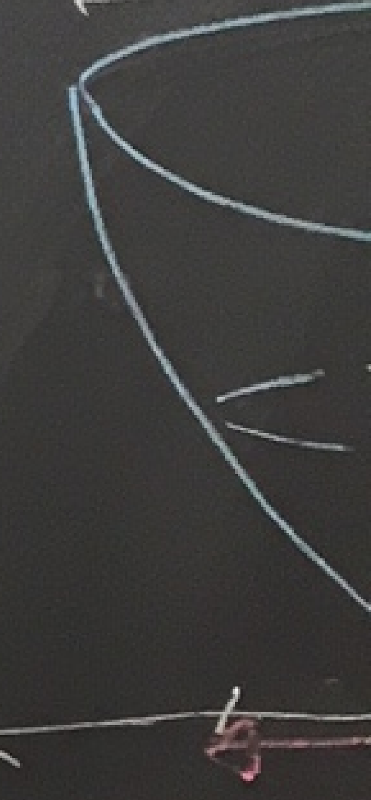
Thurs
9/12

12.6 continued...

**Theorem** Let $f(x,y)$ be differentiable at $(a,b)$ with $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle \neq \langle 0,0\rangle$

1. $f$ has its maximum rate of increase at $(a,b)$ in the direction of the gradiant vector $\nabla f(a,b)$. In that direction, the rate of increase is $|\nabla f(a,b)|$
2. $f$ has its maximum rate of decrease at $(a,b)$ in the direction of $-\nabla f(a,b)$. In that direction the rate of decrease is $-|\nabla f(a,b)|$.
3. The directional derivate is $0$ in any direction perpendicular to $\nabla f(a,b)$

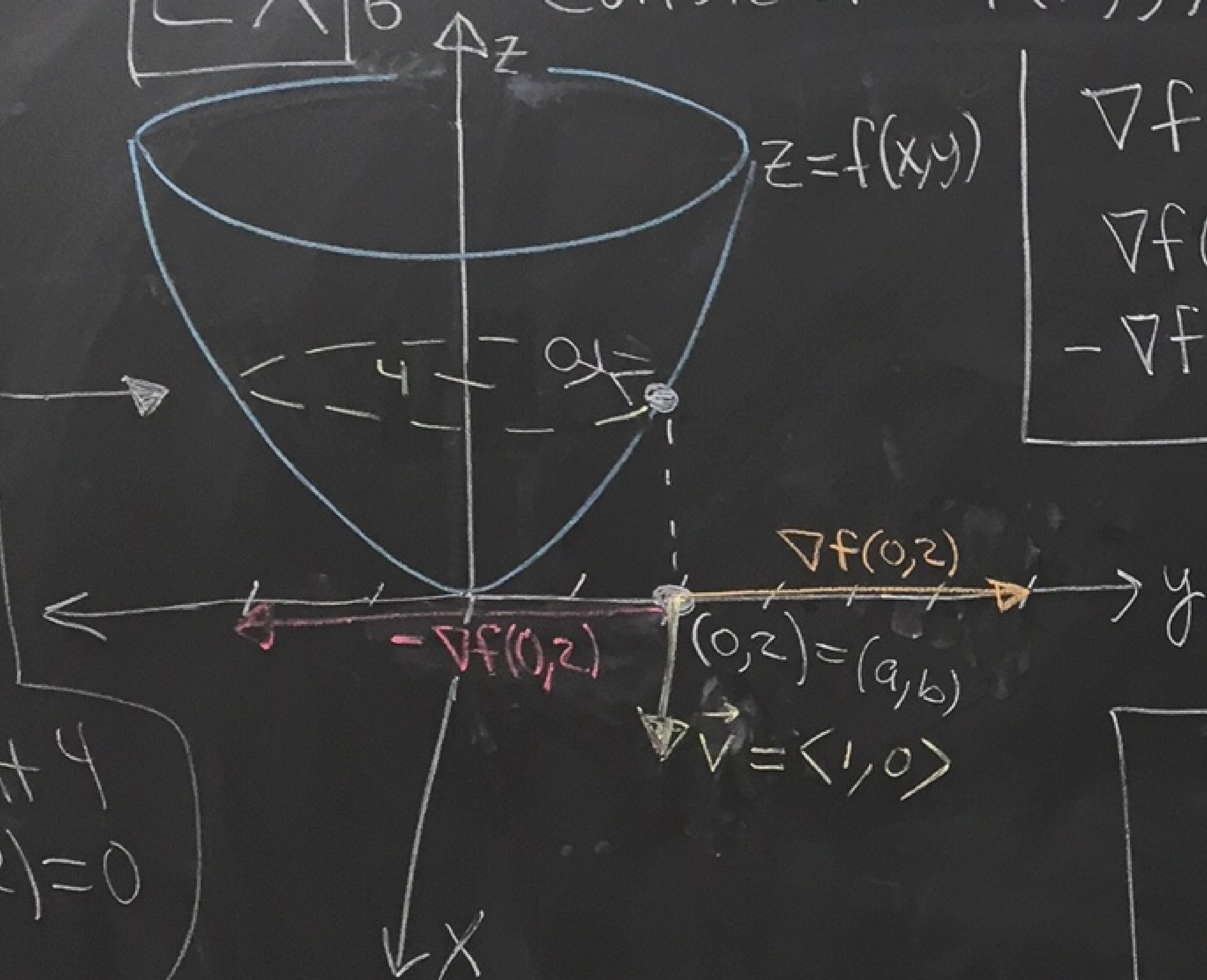
Ex



$\boxed{\text{Ex}}$: Consider $f(x,y) = x^2 + y^2$ at $(a,b) = (0,2)$

If you walk in the $\vec{v}$ direction above $(0,2)$ you stay at height 4

$D_{\vec{v}} f(0,2) = 0$




$\nabla f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$

$\nabla f(0,2) = \langle 0, 4 \rangle$

$-\nabla f(0,2) = \langle -0, -4 \rangle = \langle 0, -4 \rangle$

1. The max rate of increase (max directional derivative) at $(0,2)$ is in the direction of $\nabla f(0,2) = \langle 0,4 \rangle$. And in that direction the directional derivate has value $|\nabla f(0,2)| = |\langle 0,4 \rangle| = \sqrt{0^2+4^2} = 4$

Let's verify: unit vector in direction of $\nabla f(0,2) = \langle 0,4 \rangle$ is $\vec{u} = \frac{\langle 0,4 \rangle}{|\langle 0,4 \rangle|} = \frac{\langle 0,4 \rangle}{4} = \langle 0,1 \rangle$

$D_{\vec{u}} f(0,2) = \nabla f(0,2) \cdot \vec{u} = \langle 0,4 \rangle \cdot \langle 0,1 \rangle = 0 \cdot 0 + 4 \cdot 1$

$= 4 = |\nabla f(0,2)|$

(2) The max. rate of decrease (min. directional derivative) at $(0,2)$ is in the direction of $-\nabla f(0,2) = \langle 0,-4 \rangle$. In that direction the directional derivate equals $-|\nabla f(0,2)| = -4$.

(3) Let $\vec{v} = \langle 1,0 \rangle$. Then $\vec{v}$ is perpendicular to $\nabla f(0,2)$ since $\langle 1,0 \rangle \cdot \langle 0,4 \rangle = 1 \cdot 0 + 0 \cdot 4 = 0$.

And notice $D_{\vec{v}} f(0,2) = \nabla f(0,2) \cdot \vec{v} = \langle 0,4 \rangle \cdot \langle 1,0 \rangle = 0$

<u>Theorem</u> Given a function $f(x,y)$ that is differentiable at $(a,b)$ the tangent line to the level curve of $f$ at $(a,b)$ is perpendicular to the gradient $\nabla f(a,b)$ provided $\nabla f(a,b) \neq \vec{0}$

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<u>Ex:</u> Consider $f(x,y) = x^2 + y^2$ at $(a,b) = (0,2)$.

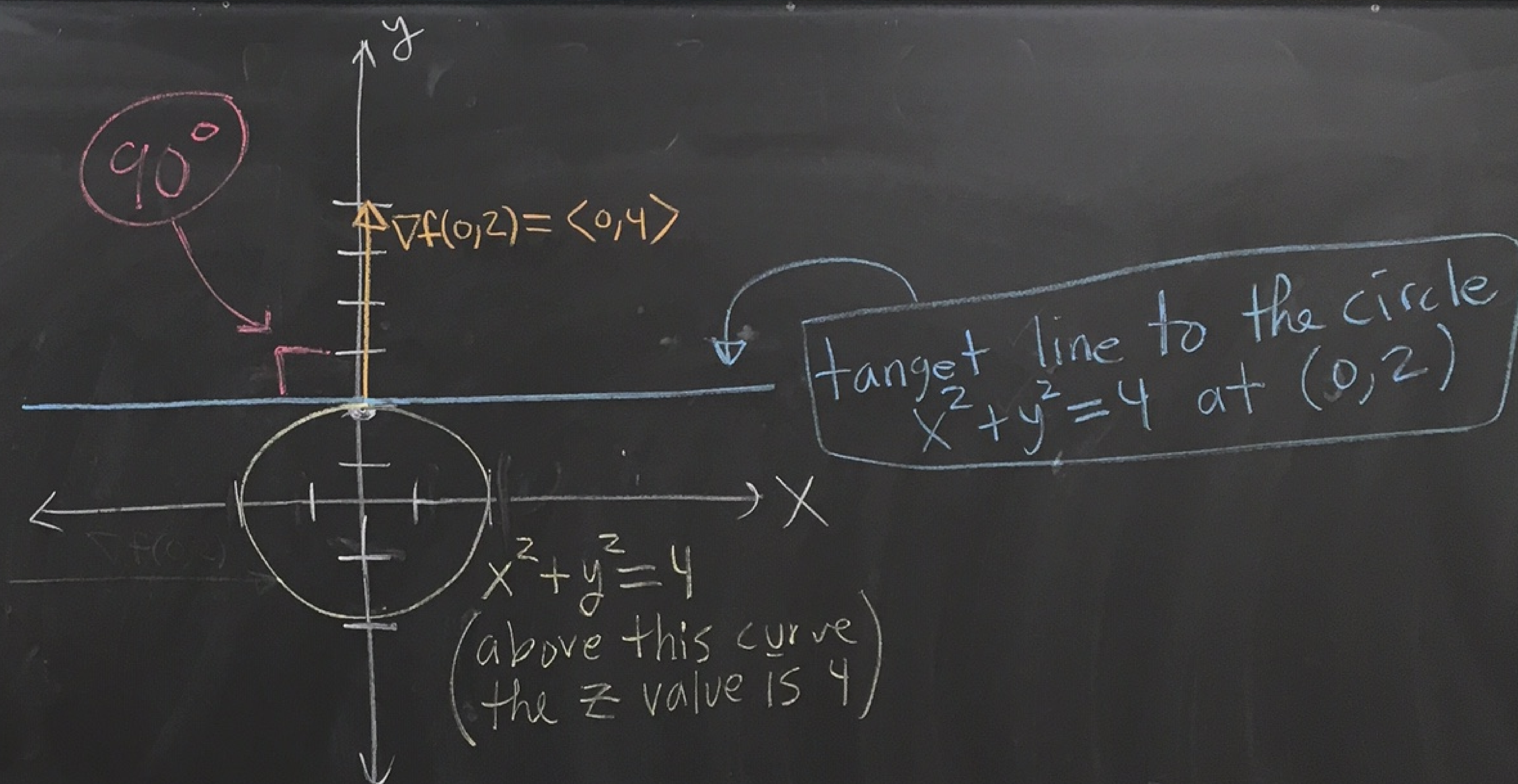
$$\nabla f(0,2) = \langle 0, 4 \rangle$$

## level curve at (0,2)

$$f(0,2) = 0^2 + 2^2 = 4$$

level curve is

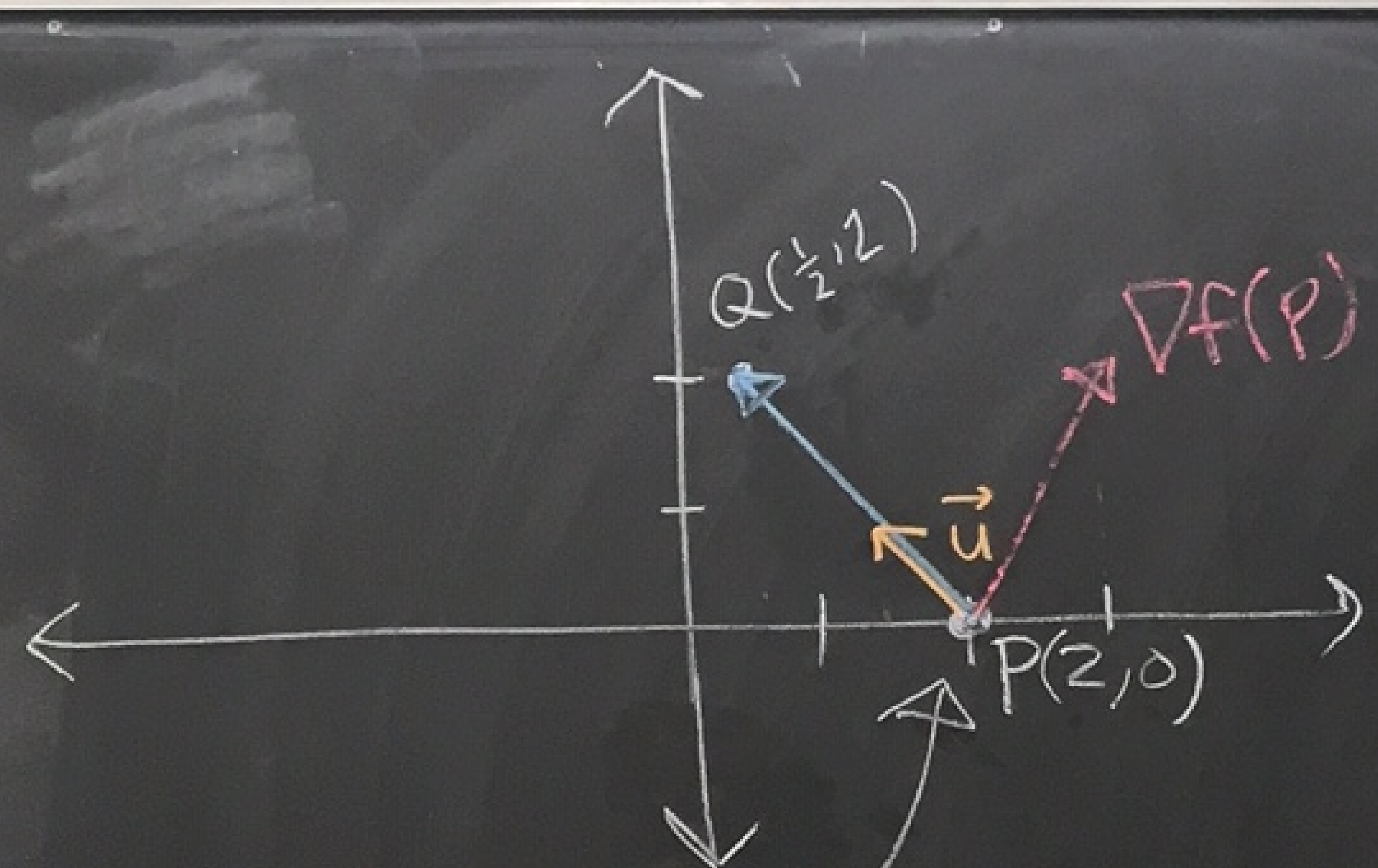
$$4 = f(x,y) = x^2 + y^2$$



Ex: Let $f(x,y) = xe^y$.

(a) Find the rate of change (directional derivate) of $f$ at the point $P(2,0)$ in the direction from $P$ to $Q(\frac{1}{2}, 2)$.

(b) In what direction does $f$ have the maximum rate of change? What is that rate of change?




At P, going in the $\vec{u}$ direction gives slope 1. Going in the $\nabla f(P)$ gives slope $\sqrt{5}$ and is the max slope at P

$$\overrightarrow{PQ} = \langle \tfrac{1}{2}-2, 2-0 \rangle = \langle -\tfrac{3}{2}, 2 \rangle$$

$$|\overrightarrow{PQ}| = |\langle -\tfrac{3}{2}, 2\rangle| = \sqrt{\left(-\tfrac{3}{2}\right)^2 + 2^2} = \sqrt{\tfrac{9}{4}+4} = \sqrt{\tfrac{9+16}{4}} = \sqrt{\tfrac{25}{4}} = \tfrac{5}{2}$$

Make a unit vector in the direction of $\overrightarrow{PQ}$.

$$\vec{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{1}{\left(\frac{5}{2}\right)}\langle -\tfrac{3}{2}, 2\rangle = \langle -\tfrac{3}{5}, \tfrac{4}{5}\rangle$$

$$\nabla f = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(P) = \nabla f(2,0) = \langle e^0, 2e^0 \rangle = \langle 1, 2 \rangle$$

(a) $D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} = \langle 1,2 \rangle \cdot \langle -\tfrac{3}{5}, \tfrac{4}{5} \rangle = (1)\left(-\tfrac{3}{5}\right) + (2)\left(\tfrac{4}{5}\right) = 1$

(b) max rate of change at $P$ is $|\nabla f(P)| = |\langle 1,2\rangle|$

$= \sqrt{1^2+2^2} = \sqrt{5}$

direction of max increase is $\nabla f(P) = \langle 1,2\rangle$