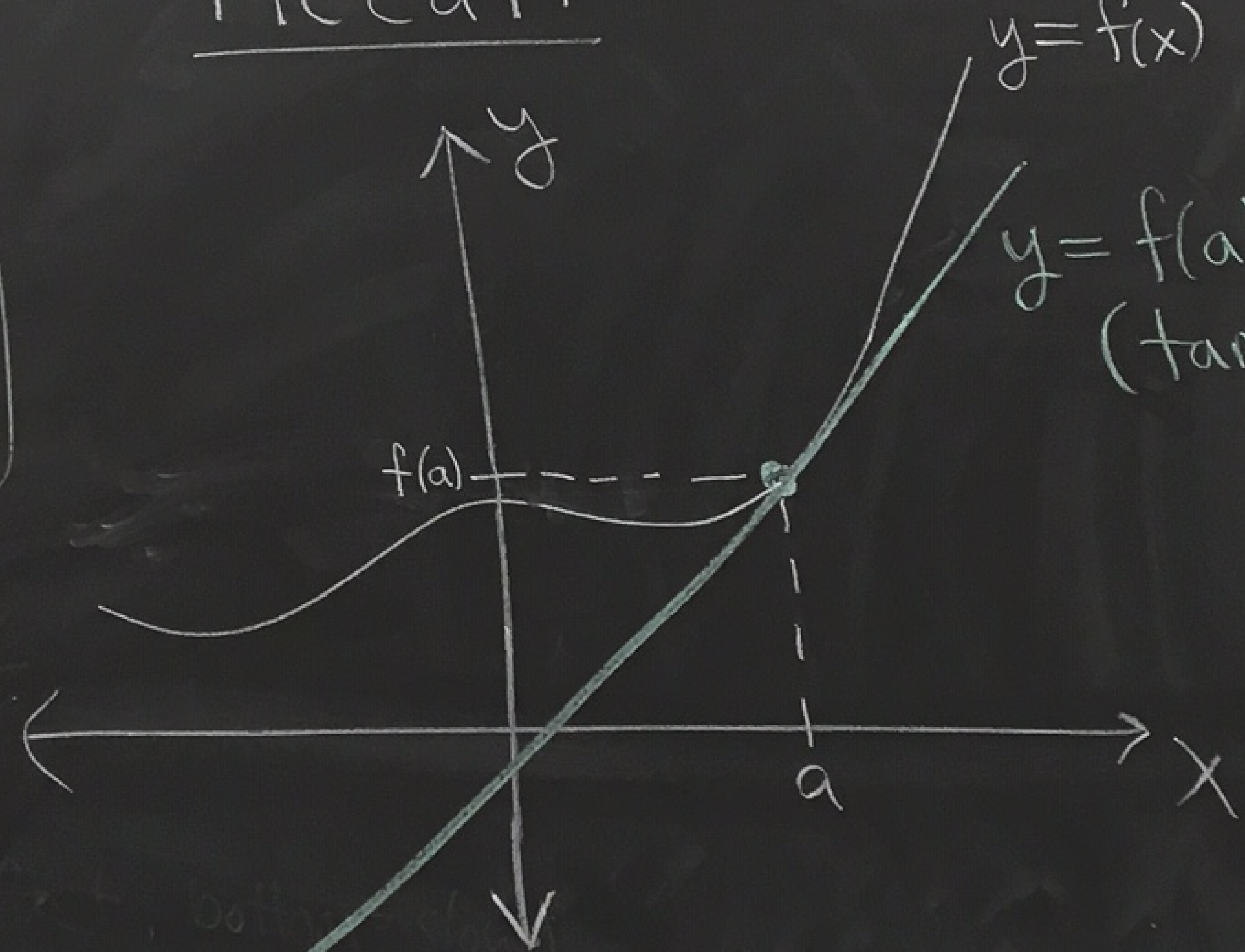
Tuesday
9/17

12.7 – Tangent planes and linear approximations

Recall

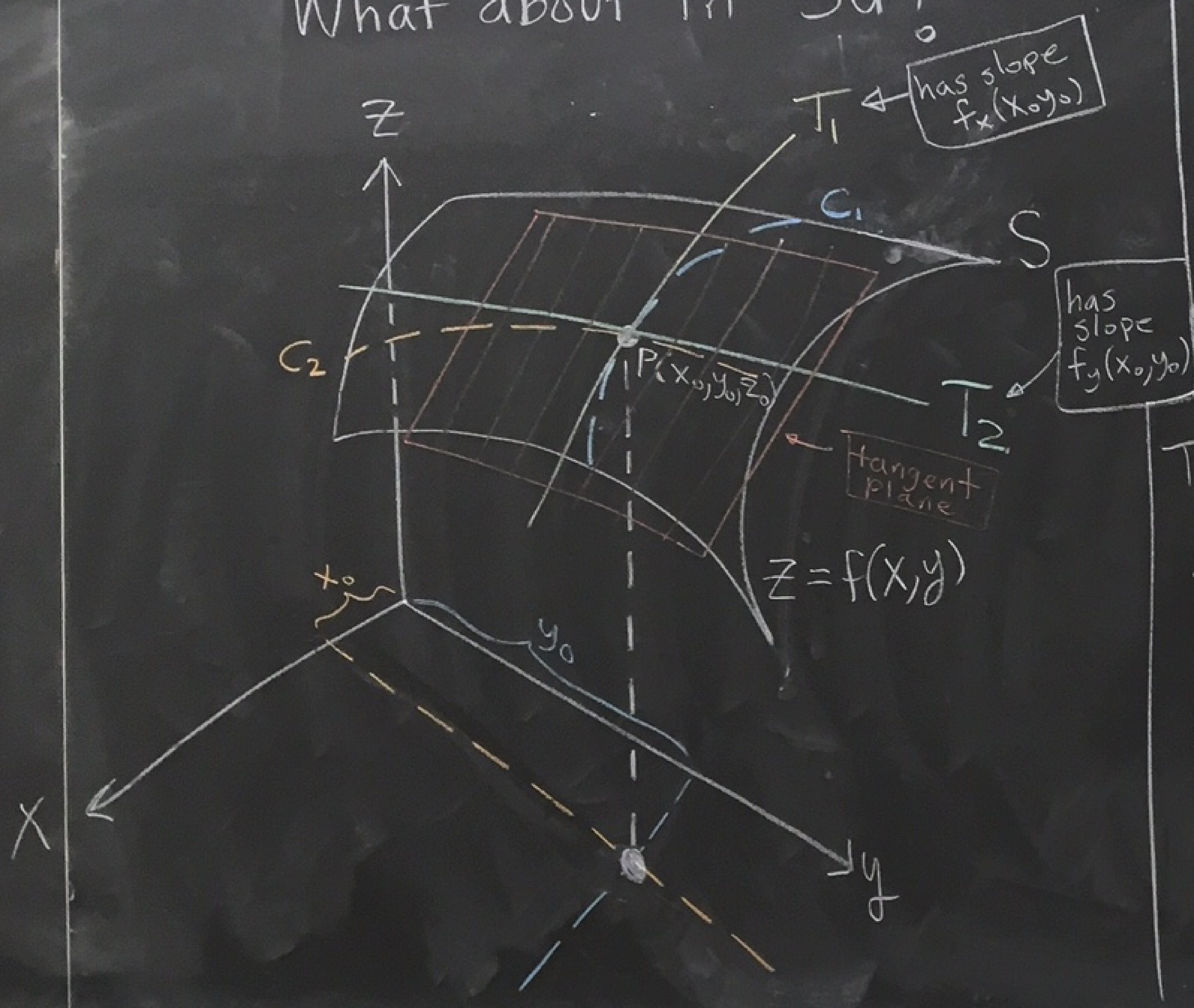
$$\boxed{\begin{aligned} y - y_0 &= m(x - x_0) \\ y - f(a) &= f'(a)(x-a) \end{aligned}}$$



$y = f(a) + f'(a)(x-a)$
(tangent line)

X

What about in 3d?



Suppose a surface S has equation $z=f(x,y)$, where f has continuous first partial derivatives f_x and f_y, and let $P(x_0,y_0,z_0)$ be a point on S.

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y=y_0$ and $x=x_0$ with S. The point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.

The <u>tangent plane</u> to the surface S at the point P is defined to be the plane that contains both T_1 and T_2.

The tangent plane has equation

$$\boxed{z-z_0=f_x(x_0,y_0)\cdot(x-x_0)+f_y(x_0,y_0)\cdot(y-y_0)}$$

Ex: Find an equation of the tangent plane that is tangent to the surface $z = 12 - 3x^2 - 4y^2$ at the point $(1,1,5)$

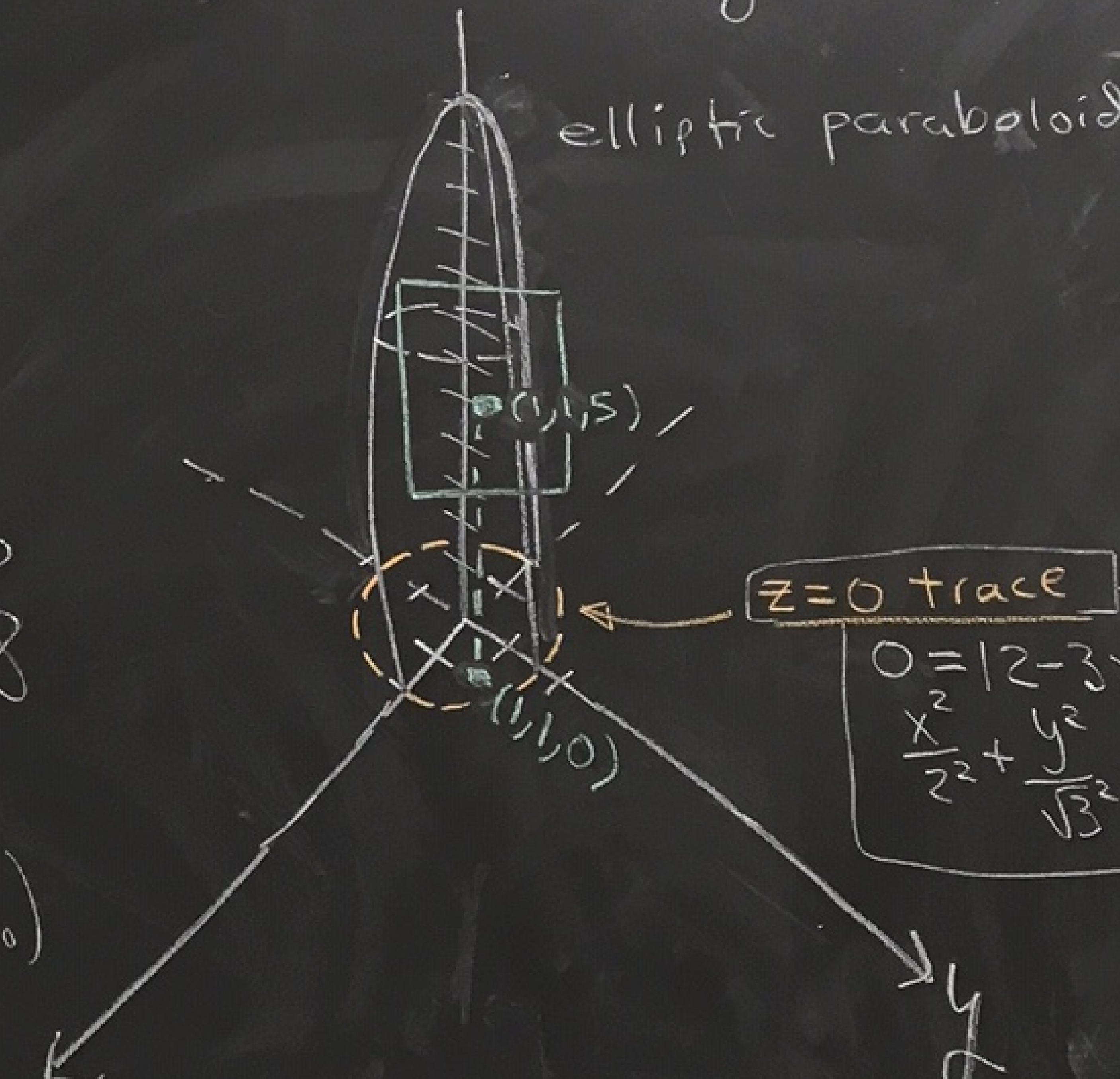
$$f(x,y) = 12 - 3x^2 - 4y^2$$

$$(x_0, y_0, z_0) = (1,1,5)$$

$$\left.\begin{aligned} f_x &= -6x \\ f_y &= -8y \end{aligned}\right\} \begin{aligned} f_x(x_0,y_0) &= f_x(1,1) = -6 \\ f_y(x_0,y_0) &= f_y(1,1) = -8 \end{aligned}$$

$$z - z_0 = f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)$$

$$\boxed{z - 5 = -6(x-1) - 8(y-1)}$$



$0 = 12 - 3x^2 - 4y^2$ ellipse w/ x-radius 2 and y-radius $\sqrt{3} \approx 1.73$

$$\frac{x^2}{2^2} + \frac{y^2}{\sqrt{3}^2} = 1$$

I'm guessing the calc. is using

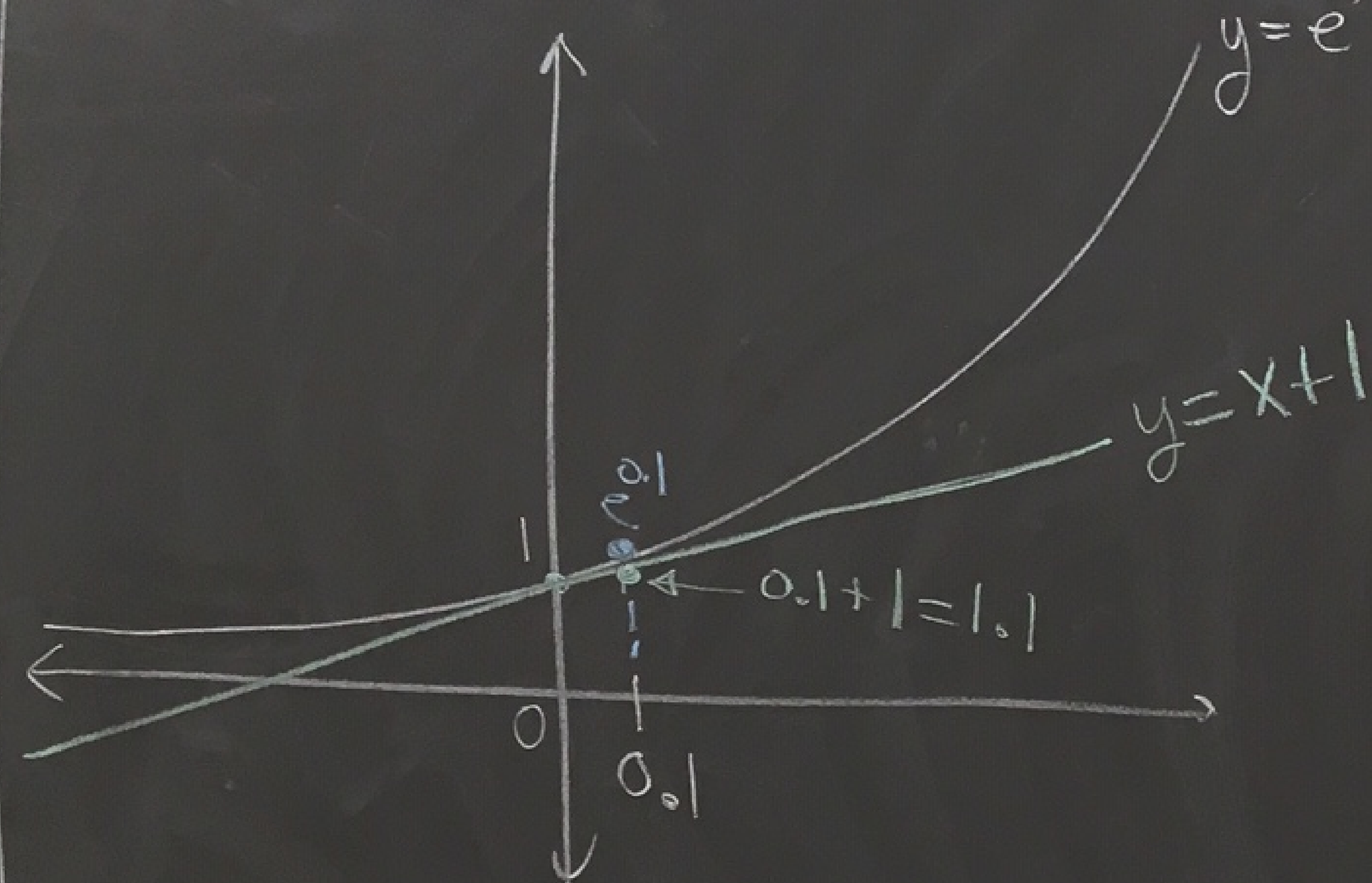
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \boxed{1+x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

linear approx

Recall from Calc I

Approximate $e^{0.1}$



Since 0.1 is close to 0, let's look at the tangent line at 0.

$f'(x)=e^x$, $(x_0,y_0)=(0,1)$

$y-1=e^0(x-0)$

$\boxed{y=x+1}$

$e^{0.1} \approx 0.1+1=1.1$

approximate with tangent line

$e^{0.1} \approx 1.015170918..$

actual approx.

The tangent line will be close to $y=e^x$ when x is near 0, ie when $x=0.1$

Linear Approximations

Given a function $f(x,y)$ with continuous first partial derivatives f_x and f_y, the linear approximation to f at (a,b) is

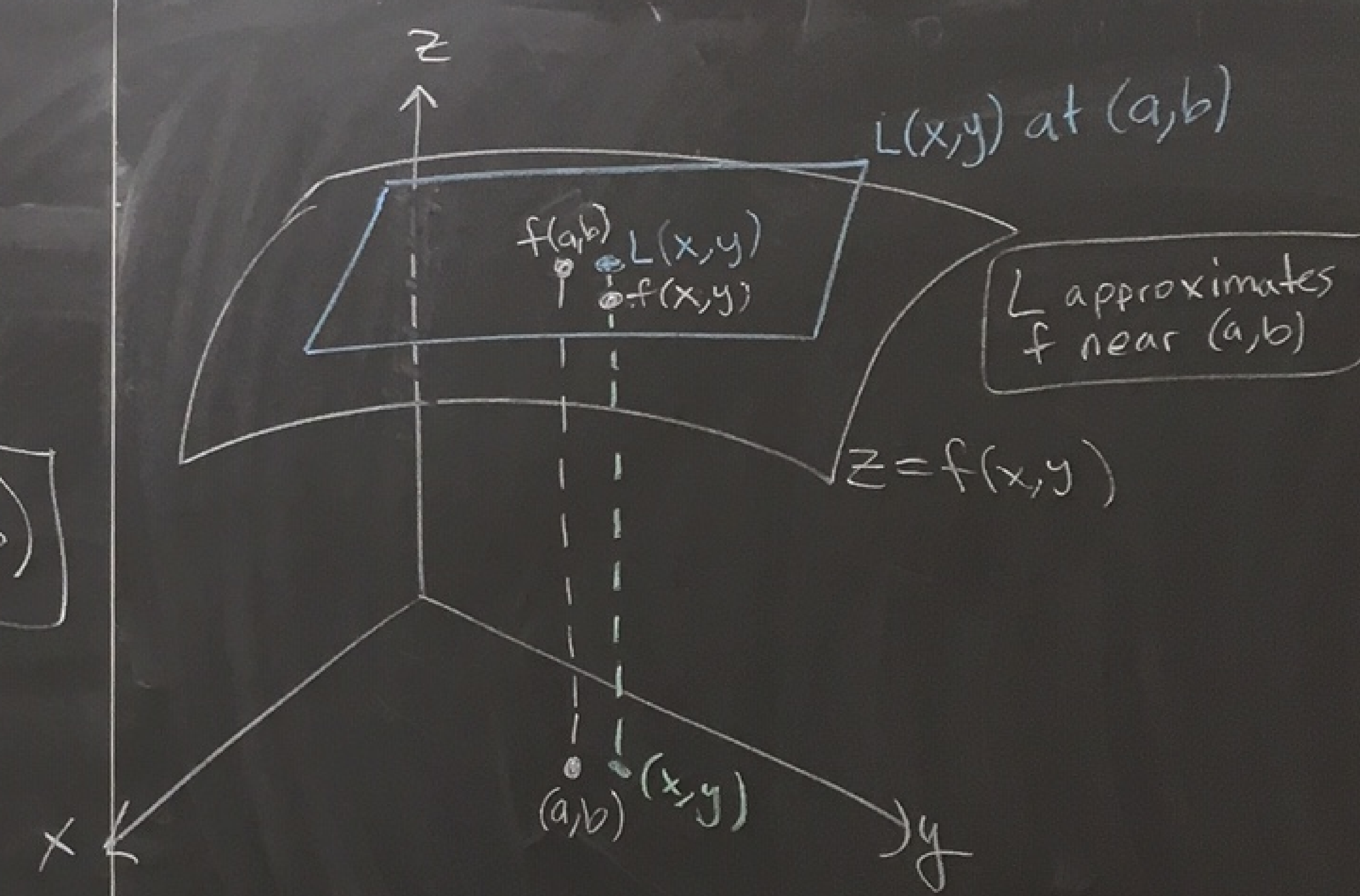
the tangent plane at (a,b)

$$\boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

MAIN IDEA

If (x,y) is "close" to (a,b) then

$$f(a,b) \approx L(a,b)$$



imates
(a,b)

Ex: Approximate $f(x,y) = xe^{xy}$
at $(x,y) = (1.1, -0.1)$

Let $(a,b) = (1,0)$. So, (a,b) is close to $(1.1, -0.1)$

Find $L(x,y)$ for f at (a,b).

$$f_x = 1 \cdot e^{xy} + x \cdot e^{xy} \cdot y = e^{xy} + xye^{xy}$$

$$f_y = x \cdot e^{xy} \cdot x = x^2 e^{xy}$$

$$f_x(1,0) = e^0 + 0 = 1$$

$$f_y(1,0) = 1 \cdot e^0 = 1$$

$$f(1,0) = 1 \cdot e^0 = 1$$

$$L(x,y) = 1 + 1 \cdot (x-1) + 1 \cdot (y-0)$$

$$L(x,y) = x + y$$

Actual approx.
$1.1e^{(1.1)(-0.1)} \approx 0.985417...$

$$(1.1)e^{(1.1)(-0.1)} = f(1.1, -0.1) \approx L(1.1, -0.1) = 1.1 - 0.1 = \boxed{1}$$

answer