Ex: $f(x,y) = x^2 e^{\cos(x+3y^4)}$

$$\frac{\partial f}{\partial x} = f_x = (2x)e^{\cos(x+3y^4)} + \left(e^{\cos(x+3y^4)} \cdot \frac{\partial}{\partial x}\cos(x+3y^4)\right)x^2$$

$$= (2x)e^{\cos(x+3y^4)} + x^2 e^{\cos(x+3y^4)} \cdot \left[-\sin(x+3y^4)\cdot(1)\right]$$

$$\frac{\partial f}{\partial y} = f_y = x^2 e^{\cos(x+3y^4)} \cdot \left[-\sin(x+3y^4)\cdot\left[0+12y^3\right]\right].$$

Thursday, 9/5

More than one variable

In general, if $f(x_1, x_2, \ldots, x_n)$ is a function of n variables, then it's partial derivative with respect to x_i is

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_i + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{h}$$

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to compute it treat all other variables as constants.

Recall from Calc I

$(gh)' = g'h + h'g$

$\left(\frac{g}{h}\right)' = \frac{g'h - h'g}{h^2}$

$[g(h(t))]' = g'(h(t)) \cdot h'(t)$

$, x_i, \dots, x_n)$

Ex: $f(x,y,z) = z^2 x^{-4} + 2z\cos\left(\frac{x+z}{z^2}\right)$

Find $\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial z} = (x^{-4})(2z) + (2)\cos\left(\frac{x+z}{z^2}\right) + (2z)\left[-\sin\left(\frac{x+z}{z^2}\right) \cdot \left[\frac{(1)(z^2) - (2z)(x+z)}{(z^2)^2}\right]\right]$$

Notation If $z = f(x,y)$, then we may differentiate f_x and f_y again to get the second partial derivatives:

f_{xy}

$\frac{\partial^2 f}{\partial y\,\partial x}$

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

$\underline{\text{Ex:}}$ $f(x,y) = 5x^2y^3 - 10x^2$

$f_x = (5y^3)(2x) - 20x = 10xy^3 - 20x$

$f_y = (5x^2)(3y^2) = \boxed{15x^2}y^2$

$\dfrac{\partial^2 f}{\partial y \partial x}$ $\xleftarrow{\quad}$

f_{xy} $\xrightarrow{\quad}$

$f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x} = (10x)(3y^2) = 30xy^2$

$f_{yx} = \dfrac{\partial^2 f}{\partial x \partial y} = (15y^2)(2x) = 30xy^2$

$\boxed{\dfrac{\partial}{\partial x}\dfrac{\partial}{\partial y}}$

the same

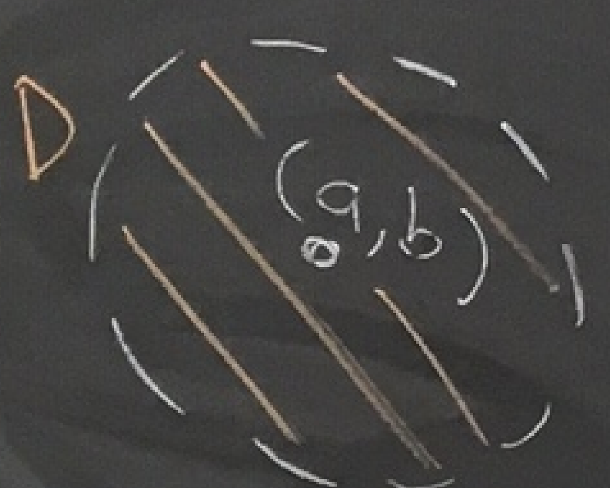
$\boxed{f_{xy} = f_{yx}}$

$f_{xx} = \dfrac{\partial^2 f}{\partial x^2} = 10y^3 - 20$

$f_{yy} = \dfrac{\partial^2}{\partial y^2} = 30x^2y$

Clairaut's Theorem

Suppose $f(x,y)$ is defined on a disk D that contains the point (a,b). If f_{xy} and f_{yx} both exist and are continuous on D



then $f_{xy}(a,b)=f_{yx}(a,b)$.

You can take even higher order partials.

$$f(x,y)=5x^2y^3-10x^2$$

$$f_{xxy}=\frac{\partial^3 f}{\partial y\,\partial x\,\partial x}=(f_{xx})_y=30y^2$$

12.5 – The Chain rule

The book uses the term "f is differentiable"
There's a technical def. of this in the book.

<u>Theorem:</u> Let $f(x,y)$ be a function of two variables. If f_x and f_y exist in a disk around (a,b) and are continuous at (a,b) then f is differentiable at (a,b).

The chain rule to rule them all

Suppose $f(x_1, x_2, \dots, x_n)$ is a differentiable function of n-variables and each $x_j = x_j(t_1, t_2, \dots, t_m)$ is a differentiable function of m-variables. Then

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

where $i = 1, 2, \dots, m$.

Ex: $f(x,y) = x^2 y + 3xy^4$

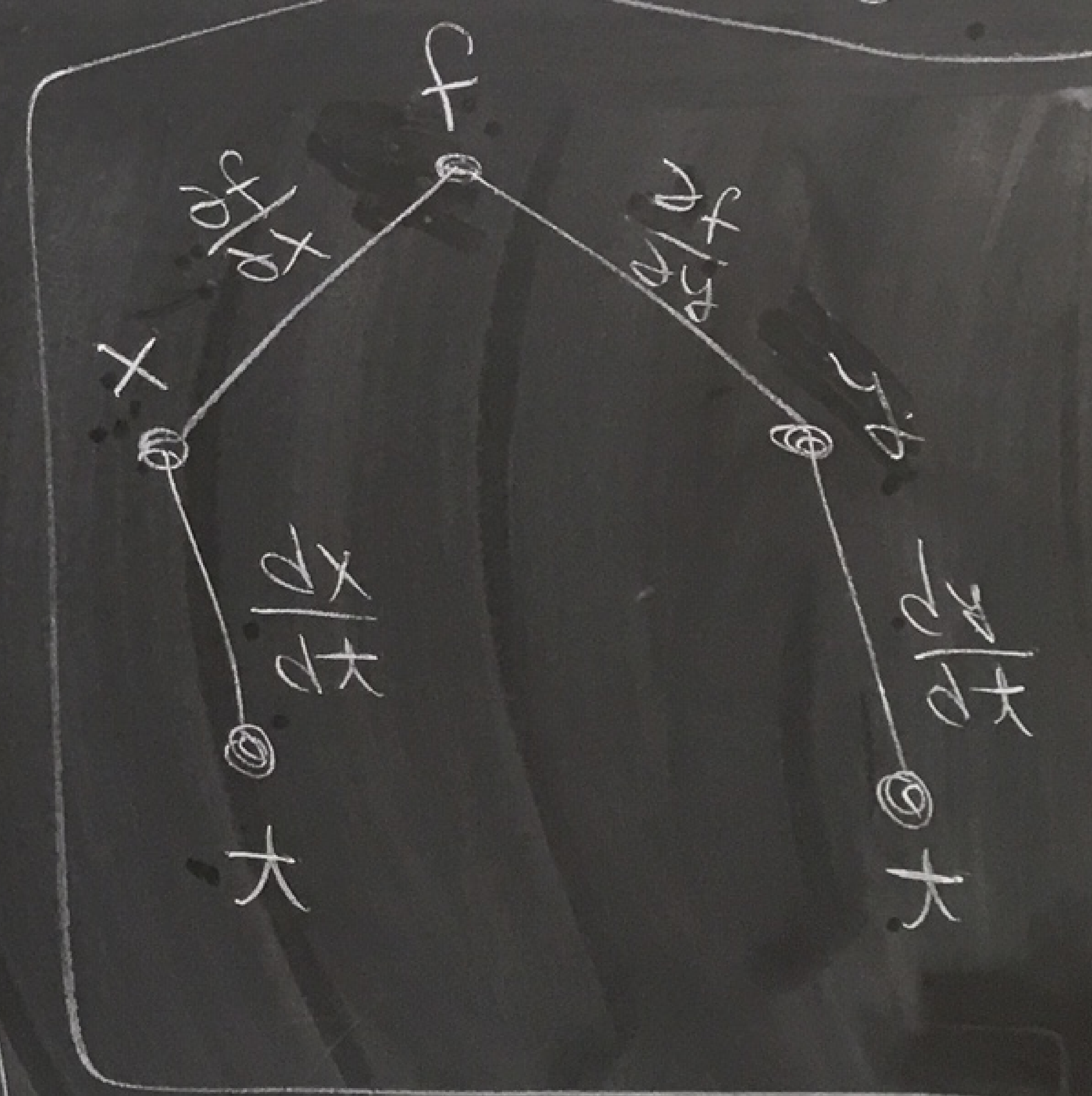
$x = \sin(2t)$

$y = \cos(t)$

Find $\frac{df}{dt}$.

Chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$



Note

$$f(t) = x^2 y + 3xy^4$$
$$= (\sin(2t))^2(\cos(t)) + 3(\sin(2t))(\cos(t))^4$$

You could just differentiate to find $\frac{df}{dt}$.

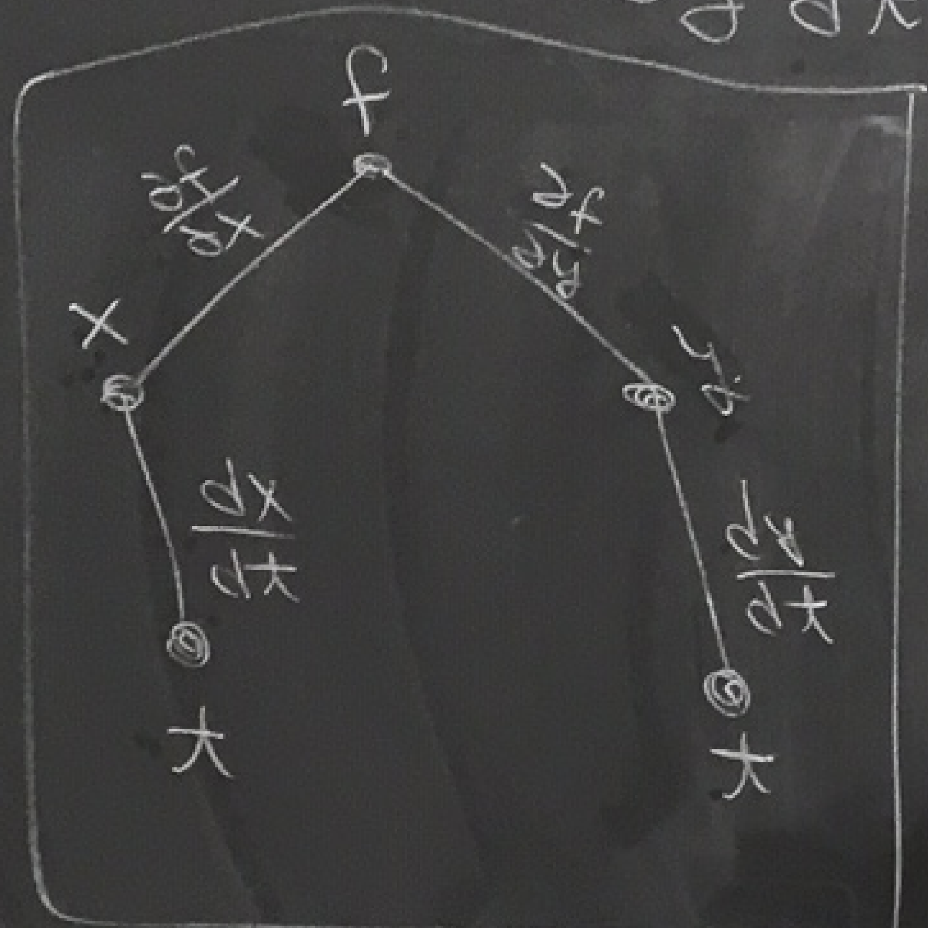
4

$(\cos(t))^4$

nd $\frac{df}{dt}$.

Chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$



$$= \overbrace{\left[y(2x) + (3y^4)(1)\right]}^{\partial f/\partial x} \cdot \overbrace{\left[2\cos(2t)\right]}^{dx/dt} + \overbrace{\left[x^2(1) + (3x)(4y^3)\right]}^{\partial f/\partial y} \overbrace{\left[-\sin(t)\right]}^{dy/dt}$$

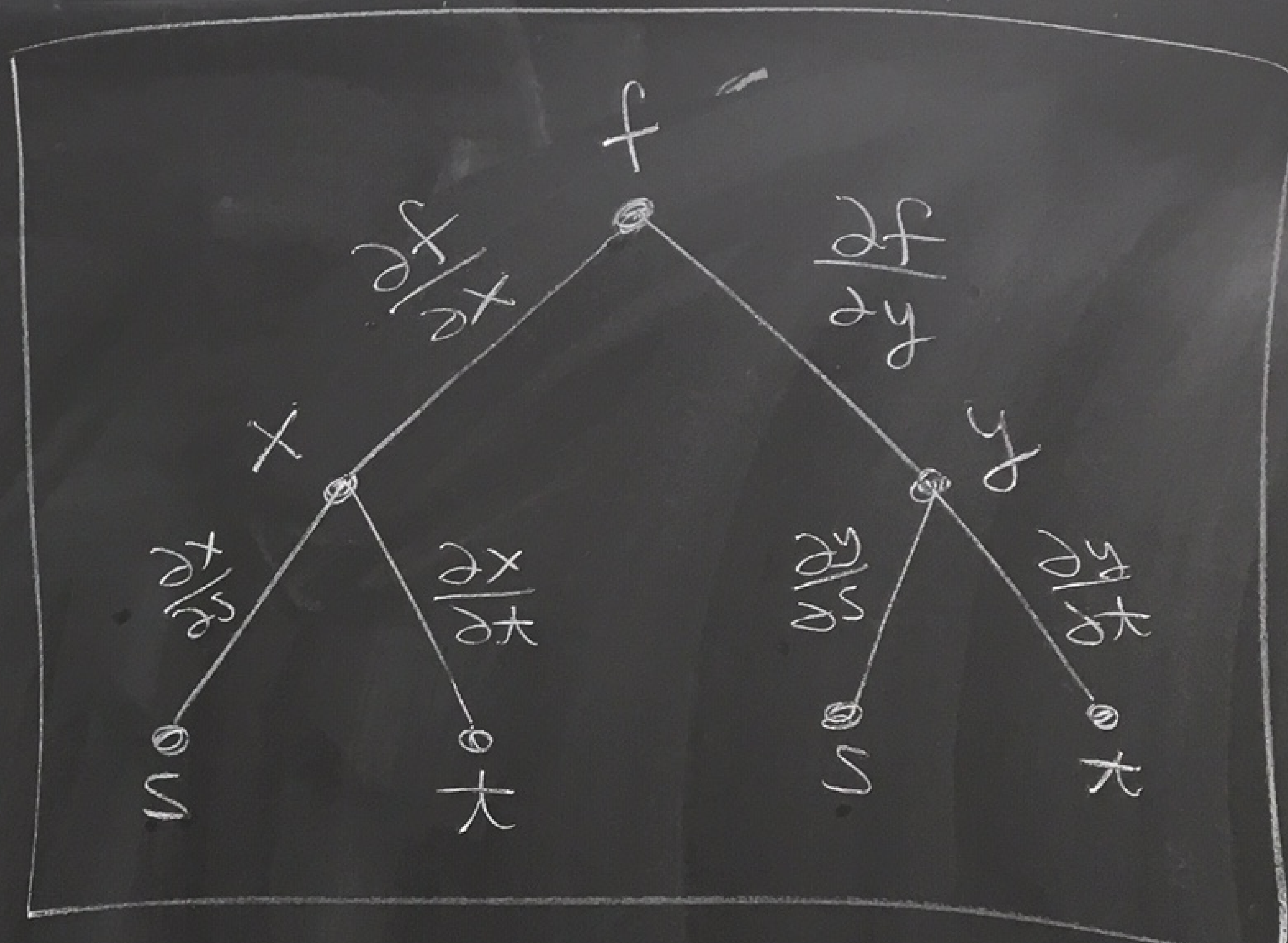
$$= \left[2\cos(t)\sin(2t) + 3\cos^4(t)\right] 2\cos(2t) + \left[\sin^2(2t) + 12\sin(2t)\cos^3(t)\right]\left[-\sin(t)\right]$$

Ex: $f(x,y) = x^2 y$

$$x = s^2 + t^2$$

$$y = 2st$$

Find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$