# ALGEBRA COMPREHENSIVE EXAMINATION

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Directions: *Answer 5 questions only.* You must answer *at least one* from each of linear algebra, groups, synthesis. Indicate CLEARLY which problems you want us to grade—otherwise, we will select which ones to grade, and they may not be the ones that you want us to grade. Be sure to show enough work that your answers are adequately supported.

Notation: $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}_n$ is the set of integers modulo $n$ and $\mathbb{C}$ is the set of complex numbers. $GL_n(\mathbb{F})$ is the group of all invertible $n \times n$ matrices with entries in the field $\mathbb{F}$ under matrix multiplication.

## Linear Algebra

(L1) Let $V$ be a vector space over a field $\mathbb{F}$. Let $W$ be a subspace of $V$. Fix an element $v_0 \in V$. Define the set $W_0 = \{v_0 + w : w \in W\}$. Prove that $W_0$ is a subspace of $V$ if and only if $v_0 \in W$.

(L2) Let $V$ be the real vector space of real functions spanned by $\{1, x, e^x\}$, and let $h : V \to V$ be defined by $h(f) = f - f'$ for all $f \in V$, that is, $h(f)$ is $f$ minus its derivative. No need to prove that $h$ is a linear function.

(1) What is the dimension of $V$?
(2) Find a basis for the space $\ker h = \{f \in V : h(f) = 0\}$.
(3) Find a basis for the space $\operatorname{im} h = \{h(f) : f \in V\}$.

(L3) Let $T$ be an arbitrary linear operator on a vector space $V$, and let $\lambda$ and $\mu$ be two distinct eigenvalues of $T$.

(a) Prove or disprove: If $v$ is an eigenvector of $T$ with eigenvalue $\lambda$, and $w$ is an eigenvector of $T$ with eigenvalue $\mu$, then $v + w$ is an eigenvector of $T$.
(b) Prove or disprove: If $v$ and $w$ are eigenvectors of $T$ with eigenvalue $\lambda$, then $v + w$ is an eigenvector of $T$.

## Groups

(G1) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_6$ and $H = \langle (2, 2) \rangle$. What familiar group is $G/H$ isomorphic to?

(G2) Let $G$ be a group. For $x, y \in G$, recall that $x$ is *conjugate* to $y$ in $G$ if there exists $g \in G$ such that $y = gxg^{-1}$. We will write $x \sim y$ to denote that $x$ is conjugate to $y$ in $G$. Prove that the relation "$\sim$" is an equivalence relation on $G$.

(G3) Let $G$ be an Abelian group. Let $H$ be the set of elements of $G$ with finite order; that is, $H = \{g \in G : |g| < \infty\}$. Prove that $H$ is a subgroup of $G$.

## Synthesis

(S1) Find a subgroup of $GL_2(\mathbb{C})$ that is isomorphic to $\mathbb{Z}_4$.

(S2) Let $H$ be the set of all matrices in $GL_2(\mathbb{R})$ with integer entries. Is $H$ a subgroup of $GL_2(\mathbb{R})$? Prove your answer.

(S3) Let $n$ be a nonnegative integer. Recall that $\mathbb{R}^*$ is the group of non-zero real numbers under multiplication. Define $\phi : GL_n(\mathbb{R}) \to \mathbb{R}^*$ by $\phi(A) = \det(A)$, the determinant of $A$.

(a) Use a property of determinants to explain why $\phi$ is a homomorphism.
(b) Determine $\ker(\phi)$ and $\operatorname{im}(\phi)$.
(c) Based on your answers to (a) and (b), use the First Isomorphism Theorem to make a statement involving a quotient group of $GL_n(\mathbb{R})$.