California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis Fall 2023

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Fall 2023 #1. Consider the sequence defined by $x_0 = 2$ and

$$x_{n+1} = 2 + \frac{2}{x_n}$$

for all integers $n \geq 0$.

(a) Show that the sequence satisfies

$$2 \leq x_n \leq 4$$

for all non-negative integers n.

(b) Prove that (x_n) has a convergent subsequence (x_{n_k}). Hint: Use your answer from (a).

Fall 2023 #2.

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$. Let $a \in \mathbb{R}$. Using the variables ϵ and δ, give the precise definition of what it means for f to be continuous at a.

(b) Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Is f continuous at 0? Use your definition from (a) to prove that your answer is correct.

Fall 2023 #3. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in $\mathbb{R}$. Show that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Fall 2023 #4. Let H be a Hilbert space. Suppose that H is the orthogonal direct sum of two closed subspaces M and N. Moreover, suppose that E is an orthonormal basis for M, and suppose that F is an orthonormal basis for N. Prove that $E \cup F$ is an orthonormal basis for H.

Fall 2023 #5. Let M be a vector space equipped with a norm $\|\cdot\|_{\mathrm{M}}$.

(a) Show that $d_{\mathrm{M}}(x,y) = \|x-y\|_{\mathrm{M}}$ is a metric on M.
(b) Prove that there exists a metric d on a vector space M such that there does not exist a norm $\|\cdot\|_{\mathrm{M}}$ on M with $d(x,y) = \|x-y\|_{\mathrm{M}}$ for all $x, y \in$ M. (In other words, prove by means of a counterexample that not every metric on M is induced by a norm.)

Fall 2023 #6. Let P be the vector space of all polynomials with complex coefficients of degree zero or one, equipped with the inner product $\langle f, g\rangle = \int_0^2 f(t)\overline{g}(t)\,dt$.

(a) Find an orthonormal basis of P with respect to this inner product.
(b) Find two constants a and b which minimize the quantity

$$I = \int_0^2 |t^3 - a - bt|^2\,dt.$$

Fall 2023 #7. Let $C([0,1];\mathbb{C})$ be the normed vector space of continuous functions from $[0,1]$ to $\mathbb{C}$. Take the L^∞ norm on $C([0,1];\mathbb{C})$, that is, $||f||_\infty = \max\{|f(x)| \,:\, 0 \leq x \leq 1\}$.

Define $T\colon C([0,1];\mathbb{C}) \to C([0,1];\mathbb{C})$ by $(Tf)(t) = \displaystyle\int_0^t f(x)\,dx$.

You may assume without proof that T is a well-defined linear mapping.

(a) Compute $||T||$, the operator norm of T.

(b) Prove that T is continuous.