California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis   Fall 2023

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Fall 2023 #1. Consider the sequence defined by $x_0 = 2$ and

$$x_{n+1} = 2 + \frac{2}{x_n}$$

for all integers $n \geq 0$.

(a) Show that the sequence satisfies

$$2 \leq x_n \leq 4$$

for all non-negative integers n.

(b) Prove that (x_n) has a convergent subsequence (x_{n_k}). Hint: Use your answer from (a).

SOLUTION:

(a) We prove this by induction. For the $n = 0$ case, we have

$$x_0 = 2,$$

so the result obviously holds for $n = 0$. Assume now that the result holds for $n = k$, so that

$$0 \leq \frac{2}{x_k} \leq 2.$$

It follows that

$$2 \leq 2 + \frac{2}{x_k} \leq 4,$$

from which we deduce that

$$2 \leq x_{k+1} \leq 4.$$

Thus, the result holds for the $n = k + 1$ case. By the principle of mathematical induction, it follows that the result holds for all $n \geq 0$.

(b) Since the sequence is bounded, the Bolzano-Weierstrass theorem implies that there exists a convergent subsequence x_{n_k}.

Fall 2023 #2.

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$. Let $a \in \mathbb{R}$. Using the variables ϵ and δ, give the precise definition of what it means for f to be continuous at a.

(b) Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Is f continuous at 0? Use your definition from (a) to prove that your answer is correct.

(a) See, for example, Def. 3.1 here:

`https://www.math.ucdavis.edu/~hunter/m125a/intro_analysis_ch3.pdf`

(b) The function f is not continuous at 0. We will prove this by contradiction. Temporarily assume that f is continuous at 0. Then for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - 0| < \delta$, then $|f(x) - f(0)| < \epsilon$.

Because we know this statement holds for all $\epsilon > 0$, in particular, it holds when we let $\epsilon = 1$. So we may choose $\delta > 0$ such that if $|x - 0| < \delta$, then $|f(x) - f(0)| < 1$. In other words, if $|x| < \delta$, then $|f(x) - 1| < 1$, since $f(0) = 1$. In particular, for $x = -\delta/2$, we have that $|x| = \delta/2 < \delta$, and so it should be true that $|f(-\delta/2) - 1| < 1$. But $\delta > 0$, so $-\delta/2 < 0$, so $f(-\delta/2) = 0$. So $|f(-\delta/2) - 1| = |0 - 1| = 1$, which means that the statement $|f(-\delta/2) - 1| < 1$ is false.

This is a contradiction. Therefore our original assumption that that f is continuous at 0 must be false. In other words, we have proved that f is not continuous at 0.

Fall 2023 #3. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in $\mathbb{R}$. Show that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

Solution: Recall that the definition of the limit superior is

$$\limsup x_n = \lim_{n\to\infty} \sup_{m\geq n} x_m.$$

Fix $n \in \mathbb{N}$. Then

$$x_n \leq \sup_{m\geq n} x_m \quad \text{and} \quad y_n \leq \sup_{m\geq n} y_m,$$

so that

$$x_n + y_n \leq \sup_{m\geq n} x_m + \sup_{m\geq n} y_m.$$

Note that the right-hand side is an upper bound for $x_n + y_n$ whenever $n \leq m$, which must be greater than the least upper bound. Thus, we see that

$$\sup_{m\geq n}(x_m + y_m) \leq \sup_{m\geq n} x_m + \sup_{m\geq n} y_m$$

Taking the limit yields

$$\lim_{n\to\infty} \sup_{m\geq n}(x_m + y_m) \leq \lim_{n\to\infty} \left(\sup_{m\geq n} x_m + \sup_{m\geq n} y_m \right).$$

Next, we note that the sequences

$$\overline{x}_n = \sup_{m\geq n} x_m$$

is monotone decreasing and bounded, so that the limits

$$\lim_{n\to\infty} \sup_{m\geq n} x_m \quad \text{and} \quad \lim_{n\to\infty} \sup_{m\geq n} y_m$$

exist. Thus

$$\begin{aligned}
\limsup(x_n + y_n) &= \lim_{n\to\infty} \sup_{m\geq n} (x_m + y_m) \\
&\leq \lim_{n\to\infty} \left(\sup_{m\geq n} x_m + \sup_{m\geq n} y_m \right) \\
&= \lim_{n\to\infty} \left(\sup_{m\geq n} x_m \right) + \lim_{n\to\infty} \left(\sup_{m\geq n} y_m \right) \\
&= \limsup x_n + \limsup y_n.
\end{aligned}$$

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Fall 2023 #4. Let H be a Hilbert space. Suppose that H is the orthogonal direct sum of two closed subspaces M and N. Moreover, suppose that E is an orthonormal basis for M, and suppose that F is an orthonormal basis for N. Prove that $E \cup F$ is an orthonormal basis for H.

Solution:

See (3) on p. 4 here:

`https://drive.google.com/file/d/1nJOpjo3RLl-LHZquBTLlemthhKPYzarY/view?pli=1`

Fall 2023 #5. Let M be a vector space equipped with a norm $\|\cdot\|_{\mathrm{M}}$.

(1) Show that $d_{\mathrm{M}}(x, y) = \|x - y\|_{\mathrm{M}}$ is a metric on M.
(2) Prove that there exists a metric d on a vector space M such that there does not exist a norm $\|\cdot\|_{\mathrm{M}}$ on M with $d(x, y) = \|x - y\|_{\mathrm{M}}$ for all $x, y \in \mathrm{M}$. (In other words, prove by means of

a counterexample that not every metric on M is induced my a norm.)

Solutions:

(1) See:

https://proofwiki.org/wiki/Metric_Induced_by_Norm_is_Metric

(2) See Exercise 2(i) here:

https://www.uio.no/studier/emner/matnat/math/MAT2400/v20/solutions/solutions_week12.pdf

Fall 2023 #6. Let P be the vector space of all polynomials with complex coefficients of degree zero or one, equipped with the inner product $\langle f, g\rangle = \int_0^2 f(t)\overline{g}(t)\,dt$.

(1) Find an orthonormal basis of P with respect to this inner product.
(2) Find two constants a and b which minimize the quantity

$$I = \int_0^2 |t^3 - a - bt|^2\,dt.$$

Solutions:

(1) Note that $\{1, t\}$ is an ordered basis of P. We apply the Gram-Schmidt process:

$v_1 = 1/||1||$

$$||1|| = \sqrt{\int_0^2 1^2\,dt} = \sqrt{2}$$

So $v_1 = 1/\sqrt{2}$.

Then take

$$v_2 = \frac{t - \langle t, v_1\rangle v_1}{\|t - \langle t, v_1\rangle v_1\|}.$$

If we observe that

$$t - \langle t, v_1\rangle v_1 = t - \frac{1}{\sqrt{2}}\int_0^2 \frac{1}{\sqrt{2}}t = t - 1$$

and that

$$\|t - \langle t, v_1\rangle v_1\| = \frac{2}{3},$$

we then obtain

$$v_2 = \sqrt{\frac{3}{2}}(t-1).$$

Then $\{v_1, v_2\}$ is an o.n. basis for P.

(2) This quantity will be minimized when we take the projection of t^3 onto P.

This projection equals:

$$\begin{aligned}\langle t^3, v_1\rangle v_1 + \langle t^3, v_2\rangle v_2 &= \frac{1}{\sqrt{2}}\int_0^2 \frac{1}{\sqrt{2}}t^3\, dt + \sqrt{\frac{3}{2}}(t-1)\int_0^2 \sqrt{\frac{3}{2}}t^3(t-1)\, dt \\ &= 4 + \frac{18}{5}(t-1) \\ &= \frac{18}{5}t - \frac{2}{5}.\end{aligned}$$

So we can minimize I when we take $a = -\frac{2}{5}, b = \frac{18}{5}$.

Fall 2023 #7. Let $C([0,1];\mathbb{C})$ be the normed vector space of continuous functions from $[0,1]$ to $\mathbb{C}$. Take the L^∞ norm on $C([0,1];\mathbb{C})$, that is, $||f||_\infty = \max\{|f(x)| : 0 \le x \le 1\}$.

Define $T\colon C([0,1];\mathbb{C}) \to C([0,1];\mathbb{C})$ by $(Tf)(t) = \displaystyle\int_0^t f(x)\, dx$.

You may assume without proof that T is a well-defined linear mapping.

(1) Compute $||T||$, the operator norm of T.
(2) Prove that T is continuous.

Solution:

See #2(a) here:

`https://math.berkeley.edu/~brent/files/105_hw2.pdf`

They don't say it explicitly in that proof, but the fact that $||T||$ is finite shows that T is a bounded linear operator and is therefore continuous.