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A PLATONIST EPISTEMOLOGY*

ABSTRACT. A response is given here to Benacerraf's 1973 argument that mathematical platonism is incompatible with a naturalistic epistemology. Unlike almost all previous platonist responses to Benacerraf, the response given here is positive rather than negative; that is, rather than trying to find a problem with Benacerraf's argument, I accept his challenge and meet it head on by constructing an epistemology of abstract (i.e., aspatial and atemporal) mathematical objects. Thus, I show that spatio-temporal creatures like ourselves can attain knowledge about mathematical objects by simply explaining *how* they can do this. My argument is based upon the adoption of a particular version of platonism – full-blooded platonism – which asserts that any mathematical object which possibly *could* exist actually *does* exist.

1. THE EPISTEMOLOGICAL ARGUMENT AGAINST PLATONISM

Mathematical platonism is the view that mathematical theories are descriptions of a non-physical (i.e., aspatial, atemporal, mind-independent) aspect of reality.[1] Thus, platonists believe that mathematics is about a real and objective mathematical realm. Most platonists believe that this realm is made up primarily of mathematical *objects*, such as numbers, sets, and functions; others want to resist this. I do not want to take sides here, but for convenience, I will often speak of mathematical objects.

I will not try to argue in this paper that platonism is true. Rather, I will try to fend off what is widely regarded as the best attack on platonism, namely, Paul Benacerraf's epistemological attack.[2] And I will not take the *easy* route here; that is, I will not simply find a false premise in Benacerraf's argument. On the contrary, I will provide precisely what anti-platonists want, namely, an epistemology of abstract objects, i.e., an explanation of how human beings attain knowledge of such objects.

The reason I think it is necessary to take the difficult road here is that even if Benacerraf's particular way of formulating the argument (in terms of a causal theory of knowledge) turns out to be flawed, it is quite clear that he is onto a *prima facie* worry that platonists need to address. The worry is, of course, that it is unclear how human beings – being wholly spatio-temporal creatures – could acquire knowledge of entities that exist outside of spacetime;[3] and the reason this is worrisome is that according

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to platonism, our mathematical knowledge *is* knowledge of such objects. Hartry Field has put this point in the following way:

> But special 'reliability relations' between the mathematical realm and the belief states of mathematicians seem altogether too much to swallow. It is rather as if someone claimed that his or her belief states about the daily happenings in a remote village in Nepal were nearly all disquotationally true, despite the absence of any mechanism to explain the correlation between those belief states and the happenings in the village.[4]

So it seems that we simply do not need an argument *establishing* that human beings cannot acquire knowledge of abstract objects; for given that there is a *prima facie* worry about their ability to do so, we have a *prima facie* argument against platonism. If platonists are going to make their view seem plausible, they are going to have to address this worry by providing an epistemology, i.e., by providing an explanation of how human beings acquire knowledge of abstract mathematical objects.

In addition to the fact that I will provide what anti-platonists demand – viz., an epistemology – I will also grant an assumption that they want platonists to grant, namely, that human beings do not possess any mysterious faculty of mathematical intuition that provides them with some sort of *epistemic access* to (i.e., *contact* with) the mathematical realm.[5] Thus, what I will do is explain why such contact is unnecessary for the acquisition of knowledge of mathematical objects, i.e., why human beings can acquire mathematical knowledge without checking their work (so to speak) against the mathematical facts.[6]

2. SKELETON OF THE REFUTATION OF THE EPISTEMOLOGICAL ARGUMENT

The fundamental intuition behind my refutation of the epistemological argument is quite simple. There is a particular version of platonism – which I will call *full-blooded* platonism, or FBP – that enables us to explain how human beings can acquire knowledge of the mathematical realm. FBP can be expressed very intuitively (but also very sloppily) as the view that all possible mathematical objects exist. To give a more precise formulation of the view, we need to get rid of the *de re* modality; thus, we might say that FBP is the view that all the mathematical objects which possibly *could* exist actually *do* exist, or perhaps that there exist mathematical objects of all kinds.[7] (For rhetorical reasons, I will often use the first expression of FBP, in spite of its imprecision.) The advantage of FBP is that it eliminates the mystery of how human beings could attain knowledge of mathematical objects. For if FBP is correct, then all we have to do in order to attain such knowledge is conceptualize, or think about, or even "dream up", a mathematical object. Whatever we come up with, so long

as it is consistent, we will have formed an accurate representation of *some* mathematical object, because, according to FBP, all possible mathematical objects exist.

I can formulate this as a direct response to Field's Nepal argument. I admit that I could not have knowledge of a Nepalese village without any access to it. But if all possible Nepalese villages existed, then I *could* have knowledge of these villages, even without any access to them. To attain such knowledge, I would merely have to dream up a possible Nepalese village. For, on the assumption that all possible Nepalese villages exist, it would follow that the village that I have imagined exists and that my beliefs about this village correspond to the facts about it. Now, of course, it is not the case that all possible Nepalese villages exist, and so we cannot attain knowledge of them in this way. But according to FBP, all possible mathematical objects do exist. Therefore, if we adopt FBP, we are free to also adopt this sort of epistemology for mathematical objects.

Now, despite the simplicity of the fundamental position here, the argument I will use to show that this position provides an adequate refutation of the epistemological argument against platonism is quite long and complicated. But before I state this argument, I want to make my definition of FBP a bit more precise. When I say that FBP is the view that all possible mathematical objects exist (or, more precisely, that all the mathematical objects which possibly *could* exist actually *do* exist) I use 'possible' in its broadest sense; that is, I mean *logically possible*. This guarantees that FBP is incompatible with non-full-blooded versions of platonism which deny certain sorts of mathematical objects but assert that these objects are, in some sense, "metaphysically impossible".

It follows immediately from this that according to FBP, all consistent purely mathematical theories truly describe some part of the mathematical realm (i.e., some collection of actual mathematical objects). In order to motivate this, I have to specify what I mean by 'consistent purely mathematical theory'. First, a *theory* is just a collection of sentences. (It need not be very substantive; a single atomic sentence is a theory.) Second, a sentence or theory is *purely mathematical* if it speaks of nothing but the mathematical realm, i.e., does nothing but predicate mathematical properties and mathematical relations of mathematical objects. As far as *consistency* is concerned, I will discuss this term in Section 5; for now, let me just say that I will understand it in whatever way anti-platonists decide to understand it; for instance, if they decide to follow Kreisel and take 'consistent' as a primitive, synonymous with 'logically possible', then I will too. In Section 5, I will explain why there can, in the present context, be no complaint about me taking this line.

It should now be clear why, according to FBP, all consistent purely mathematical theories are true of some part of the mathematical realm. If all possible mathematical objects exist, then any consistent purely mathematical theory will accurately describe *some* collection of these objects; if there were some such theory which did *not* do this – i.e., which spoke of a collection of objects that do *not* exist – we would have a violation of our assumption that all the mathematical objects which possibly could exist actually do exist.[8]

(It is also now apparent that FBP is similar to Resnik's view. For in spite of the fact that he takes mathematics to be about *structures* rather than objects,[9] he seems to think that all consistent purely mathematical theories describe *some* structure. Indeed, he says that a pure theory can only "be falsified by showing that it fails to characterize any pattern at all, that is, that it is inconsistent".[10] Now, Resnik never says *why* he accepts this; that is, he never formulates his position as a version of FBP (i.e., he never says that he thinks that all possible mathematical patterns exist) and he never says that it is the acceptance of FBP which leads him to think that all consistent purely mathematical theories describe some pattern; but I don't see any other way to motivate this claim except by adopting (a structuralist version of) FBP. Moreover, while Resnik recognizes that this ontological standpoint is epistemologically important – i.e., that it might be the first step toward finding a response to Benacerraf – he does not see how to properly *argue* the point.[11])[12]

I now give the skeleton of my refutation of the epistemological argument. I will do this in Fieldian terms. He writes that the challenge to platonists is to account for the fact that *if mathematicians accept p, then p*. I think this is right: what is at issue is not whether we *have* mathematical knowledge, but whether FBP-ists can *account* for this knowledge, i.e., whether they can account for how the mathematical knowledge that we *do* have could be knowledge of an inaccessible mathematical realm. Now, of course, platonists do not have to account for there being a *perfect* correlation between our mathematical beliefs and the mathematical facts; this is simply because we're human – we make mistakes. What needs to be accounted for is the fact that our mathematical beliefs are *reliable*, i.e., the fact that, *as a general rule*, if mathematicians accept a purely mathematical sentence p, then p truly describes part of the mathematical realm.[13]

Thus, I will simply try to show that FBP-ists *can* account for this fact. My argument proceeds as follows.

(1) FBP-ists can account for the fact that human beings can – without coming into contact with the mathematical realm – know of certain purely mathematical theories that they are consistent.[14] (I will justify this premise in Section 5.)

(2) If (1) is true, then FBP-ists can account for the fact that (as a general rule) if mathematicians accept a purely mathematical sentence p, then p is consistent.

((2) is fairly trivial. (1) gives us that FBP-ists can account for the fact that we have some skill at separating consistent theories from inconsistent ones. All we need to add to this in order to get (2) is that we use this skill when we are deciding what pure mathematical theories to accept and that we are fairly conservative in our acceptance of theories, i.e., that we do not accept theories when we have no clue whether they are consistent. We do not have to claim that we *never* make mistakes and accept inconsistent theories; we only need to claim that the skill alluded to in (1) makes our acceptance of a theory a somewhat reliable indicator of the consistency of the theory.[15] I do not think anyone would question this.)

(3) FBP-ists can account for the fact that (as a general rule) if mathematicians accept a purely mathematical sentence p, then p is consistent. (From (1) and (2) via modus ponens.)

(4) If FBP is true, then any consistent purely mathematical theory truly describes part of the mathematical realm. (As I made clear above, this follows from the definition of FBP.)

(5) FBP-ists can account for the fact that (as a general rule) if mathematicians accept a purely mathematical sentence p, then p truly describes part of the mathematical realm.

((5) follows pretty trivially from (3) and (4). In the present context – i.e., in the context of accounting for mathematical knowledge - FBP-ists are allowed to assume that FBP is true. But this, together with (4), gives us that all consistent purely mathematical theories truly describe part of the mathematical realm; but combining this with (3) gives us (5). Now, one might wonder *why* FBP-ists are allowed to assume here that FBP is true. The answer is that (a) in the present context, they are not trying to establish their theory – they are merely trying to account for a certain fact (viz., that we have mathematical knowledge) from *within* their theory; and (b) in general, when one is trying to show that a theory T can account for a phenomenon P, one can assume that T is true and make use of all of its resources.[16])

As I have set things up, (5) is precisely what I need. The Benacerraf/Field worry is that platonists cannot account for the reliability of our mathematical beliefs, and (5) simply asserts that FBP-ists *can* account for it. Now, the only real gap I have left in the argument for (5) is (1). I will close this gap in Section 5. But before I do that, I would like to address two sorts of worries. In the next section, I will address the worry that I haven't done enough, i.e., that (5) does not really eliminate the epistemological problem with platonism. And in Section 4, I will consider some objections to FBP; now, we have just seen that this is not really necessary, that (in the present context) I can legitimately *assume* FBP. But I want to quell the worry that I have only solved the epistemological problem with platonism by adopting an untenable version of platonism.

3. INTERNALIST VS. EXTERNALIST EXPLANATIONS

Consider the following objection to my line of argument. "All you've really explained is how it is that human beings can *stumble onto* theories that truly describe the mathematical realm. On the picture you've given us, the mathematical community accepts a mathematical theory T for a list of reasons, one of which being that T is consistent (or, more precisely, that mathematicians believe that T is consistent). Then, since FBP is true, it turns out that T truly describes part of the mathematical realm. But since mathematicians have no conception of FBP, they do not know *why* T truly describes part of the mathematical realm, and so the fact that it *does* is, in some sense, lucky. This point can also be put as follows. Let T be a purely mathematical theory which we know (or reliably believe) is consistent; (that there *are* such theories is established by (1).) Then the objection to your epistemology is that you only have an FBP-ist account of

(M1) our ability to know that *if* FBP is true, *then* T truly describes part of the mathematical realm;[17]

you do not have an FBP-ist account of

(M2) our ability to know that T truly describes part of the mathematical realm,

because you have said nothing to account for

(M3) our ability to know that FBP is true".

The problem with this objection to my epistemology is that (a) it demands an *internalist* account of the reliability of our mathematical beliefs, but (b) in order to meet the Benacerrafian epistemological challenge, platonists need only provide an *externalist* account of the reliability of our mathematical beliefs. (To give an externalist account of the reliability of S's beliefs, one merely has to explain why S's methods of belief acquisition are, *in fact*, reliable; but to give an internalist account of the reliability of S's beliefs, one must do more: one must also explain how S knows (or reliably believes) that her methods of belief acquisition are reliable. In other words, to give an internalist account, one must provide an explanation E of the reliability of S's beliefs for which we can *also* explain how S can reliably believe that E is true.)

My FBP-ist account of the reliability of our mathematical beliefs is externalist: I explain this reliability by pointing out that (a) we use our knowledge of the consistency of purely mathematical theories in fixing our purely mathematical beliefs, and (b) on the assumption that FBP is true, any method of fixing purely mathematical belief which is so constrained by knowledge of consistency is, *in fact*, reliable (i.e., any system of purely mathematical beliefs which is consistent will, in fact, truly describe part of the mathematical realm). I do not claim that actual mathematical knowers can justify FBP or even that they have any conception of FBP. Thus, what I need to argue, in order to block the above objection, is that I do not *need* mathematical knowers to have any conception of FBP, i.e., that I do not need an internalist account of mathematical knowledge in order to refute the Benacerrafian objection to platonism.

It seems *obvious* to me that platonists only need an externalist account of mathematical knowledge. We can appreciate this by reflecting on the sort of epistemological challenge that Benacerraf is trying to present, and by locating the empirical analogue of Benacerraf's challenge, i.e., the analogous challenge to our ability to attain empirical knowledge about ordinary physical objects. According to both Field and Benacerraf – and I think they are right here – it is *easy* to solve the empirical analogue of Benacerraf's challenge: we can do so by merely appealing to sense perception. But this means that Field and Benacerraf are merely demanding an externalist account of mathematical knowledge; for an appeal to sense perception can *only* provide an externalist account of our empirical knowledge; it cannot provide an internalist account. To see this, let R be some simple theory about the physical world which we could verify via sense perception (e.g., the theory that those things in Raisin Bran are purple). In internalist terms, all we can account for, by appealing to sense perception, is

(E1) our ability to know that *if* there is an external physical world of the sort that gives rise to accurate sense perceptions, *then* R is true of that world;

an appeal to sense perception does *not* yield an internalist account of

(E2) our ability to know that R is true of the physical world,

because it does nothing to explain

(E3) our ability to know that there is an external world of the sort referred to in (E1).

On the other hand, an appeal to sense perception *is* sufficient for an *externalist* account of (E2). EWA-ists – i.e., those who believe there is an external world of the sort referred to in (E1) – can give an externalist account of our empirical knowledge of physical objects by merely pointing out that (a) we use sense perception as a means of fixing our beliefs about the physical world, and (b) on the assumption that EWA is true, any method of fixing empirical belief which is so constrained by sense perception is, *in fact*, reliable. Since this is an externalist account, EWA-ists do not need to claim that actual empirical knowers can justify EWA or even that such knowers have any conception of EWA.

So the FBP-ist's situation with respect to knowledge of mathematical objects seems to be exactly analogous to the EWA-ist's situation with respect to empirical knowledge of physical objects. The FBP-ist can provide an externalist account of our mathematical knowledge that is exactly analogous to the EWA-ist's externalist account of our empirical knowledge: where the EWA-ist appeals to sense perception, the FBP-ist appeals to our ability to separate consistent theories from inconsistent theories; and where the EWA-ist appeals to EWA, the FBP-ist appeals to FBP. Moreover, the FBP-ist's attempt to provide an internalist account of mathematical knowledge and the EWA-ist's attempt to provide an internalist account of empirical knowledge break down at exactly analogous points: the former breaks down in the attempt to account for knowledge that FBP is true, and the latter breaks down in the attempt to account for knowledge that EWA is true.

It seems to me that anti-platonists can only block my argument by finding some sort of relevant disanalogy between the FBP-ist's epistemological situation and the EWA-ist's epistemological situation. They cannot allow the two situations to be analogous, because the whole point of the Benacerrafian objection is to raise a *special* problem for abstract objects, i.e., a problem which is easily solvable for physical objects. Now, there *may* be an epistemological problem – e.g., one motivated by Cartesian-style skeptical arguments – that raises a problem for both EWA and FBP; but I am simply not concerned with any such problem here; I am only concerned with the Benacerrafian worry that there is a special epistemological problem with abstract objects.[18]

The upshot of all of this is that Benacerraf's argument has to be interpreted as demanding an *externalist* account of our knowledge of mathematical objects; the anti-platonist's claim has to be that while such an account cannot be given, an externalist account of our knowledge of *physical* objects *can* be given. We cannot interpret Benacerraf as demanding an internalist account of our knowledge of mathematical objects, because this is no easier to provide for our knowledge of physical objects. (I take it that this is all entirely obvious and precisely why Benacerraf and Field formulate the demand as a demand for an externalist account of mathematical knowledge.)

The question we need to consider, then, is whether there is any relevant disanalogy between the FBP-ist's externalist account of mathematical knowledge and the EWA-ist's externalist account of empirical knowledge. I will consider two ways in which anti-platonists might try to establish such a disanalogy. The first proceeds as follows. "While it is true that most people who know things about the physical world never cognize EWA, and while it is true that, even if they did, they could not justify their assumption that EWA is true, it seems that, at *some* level, people *do* accept EWA. But the situation with respect to FBP is entirely different: people just do not assume – at *any* level – that FBP is true."

First of all, I am not sure that either of the two central claims here are right. I am not sure that people assume – at some level – that EWA is true; and if we decide to say that they do, then I do not see why we shouldn't *also* say that they assume – at some level – that FBP is true. To assume (at some level) that FBP is true is just to assume that our mathematical singular terms refer; but it seems fairly plausible to claim that this assumption is inherent (in some sense and at some level) in mathematical practice: if a mathematician comes up with a radically new (pure mathematical) theory, she can be criticized on the grounds that the theory is inconsistent or uninteresting or useless, but she cannot be criticized on the grounds that the objects of the theory do not exist. Now, criticisms of this sort *have* emerged in the history of mathematics (e.g., in connection with imaginary numbers) but, ultimately, they have never had any real effect; that is, they have never blocked the acceptance of an otherwise-acceptable theory. I think it is fair to say that, at this point in time, it is not a legitimate or interesting mathematical criticism to claim that the objects of a consistent purely mathematical theory do not exist.

But the real problem with the first attempt to establish a disanalogy between FBP and EWA is that it is irrelevant. Given that we only need an externalist account of our knowledge of mathematical objects, it simply does not matter whether anyone accepts FBP – at *any* level.[19] My claim is that people can attain knowledge of the mathematical realm – even if they do not assume (at any level) that FBP is true – by simply having a method of mathematical belief acquisition which (as a general rule) leads them to believe purely mathematical sentences and theories only if they are consistent. This is exactly analogous to the claim that people can attain knowledge of the physical world – even if they do not assume (at any level) that EWA is true – by simply looking at it with a visual apparatus which (as a general rule) depicts the world accurately. And, of course, the *reason* we can attain knowledge in these ways is that these methods of belief acquisition are, *in fact*, reliable.

A second way in which anti-platonists might try to establish a disanalogy between the externalist epistemologies of FBP-ists and EWA-ists proceeds as follows. "The FBP-ist is not on all fours with the EWA-ist, because FBP is not analogous to the bare claim that there is an external physical world; FBP states not just that there *is* an external mathematical world, but that there is a very particular *kind* of mathematical world. Because of this, your explanation of knowledge of the mathematical realm is trivial. To see why, consider an analogous explanation. Let ZFP be a version of platonism which takes Zermelo-Fraenkel set theory to be true of part of the mathematical realm. Then ZFP-ists can give an externalist explanation of our knowledge that ZF is true, because on their view, any method of belief acquisition that leads to the acceptance of ZF will be, in fact, reliable."

The problem with this argument is that it does not establish a disanalogy. For EWA is *not* the bare claim that there is an external physical world; it is the claim that there is an external physical world *of the sort referred to in (E1)*, i.e., the sort that gives rise to accurate sense perceptions, e.g., one containing photons, photon-reflecting objects, eyes, etc. It seems to me that, if anything, this is *farther* from the bare claim that there is an external physical world than FBP is from the bare claim that there is a mathematical realm. Moreover, there is also an empirical analogue to the bit about ZFP. Let QMR be a version of realism which takes quantum mechanics to be true. Then QMR-ists can give an externalist explanation of our knowledge

that QM is true, because on their view, any method of belief acquisition that leads to the acceptance of QM will be, in fact, reliable.

The externalist epistemologies of the EWA-ist and the FBP-ist are not trivial in the way that the externalist epistemologies of the ZFP-ist and the QMR-ist are. There are at least two reasons for this; I will state these reasons in terms of ZFP and FBP, but exactly analogous points could be made in terms of QMR and EWA. The first reason that the above ZFP-ist epistemology is trivial is that it does not describe a method of mathematical belief acquisition which both leads us to believe ZF *and* is reliable in general; my FBP-ist epistemology, on the other hand, does describe a method of mathematical belief acquisition which is reliable in general; indeed, it describes a *class* of such methods, viz., the class of methods which forbid the acceptance of inconsistent purely mathematical theories.[20] The second reason that the above ZFP-ist epistemology is trivial, while my FBP-ist epistemology is not, is that ZFP is a mathematical theory, whereas FBP is an ontological theory. (Unlike ZFP, which is essentially equivalent to ZF, FBP makes no claims about any *particular* mathematical objects; it merely asserts a *general* criterion for when we ought to countenance mathematical objects.) The upshot of this is that by adopting FBP, we *explain* our ability to acquire mathematical knowledge, whereas by adopting ZFP, we do no such thing, because here, mathematical knowledge is smuggled in from the start.

I can think of no other way of trying to draw a disanalogy between the epistemologies for FBP and EWA; thus, I conclude that the two epistemologies are on all fours and, therefore, that my externalist FBP-ist epistemology is sufficient to refute Benacerraf's argument.

Before going on, I want to guard against a possible misunderstanding. My intention in this section was *not* to provide a self-contained refutation of Benacerraf's argument; that is, my point is *not* that platonists do not need an epistemology for mathematical objects, because we do not have an epistemology for physical objects. On the contrary, I think we *do* have an epistemology for physical objects, namely, a perception-based externalist epistemology. This epistemology might not do everything we would like it to do, but it surely does a lot. My purpose in this section has, rather, been to argue that the FBP-based externalist epistemology that I sketched in Section 2 is on equal footing with this perception-based epistemology. It does not do everything we want of a mathematical epistemology, but it *does* do a lot; indeed, it does just as much as the perception-based epistemology does in the empirical case.

If you doubt that my explanation does a lot, consider that Benacerraf's paper is supposed to inspire an absolute befuddlement about our ability to acquire knowledge of the mathematical realm. We find ourselves asking, "How in the world could we have *any clue* about the nature of such an inaccessible realm? How could we even begin to make a *guess* in this connection?" This is decidedly different from what skeptical arguments do to us; it is entirely *obvious* how people could make correct guesses about the physical world, i.e., how they could stumble onto true hypotheses about it – they could do this by merely looking at it. All that one might wonder about is our ability to *know* things (in the skeptic's sense) about the physical world. But look what FBP does for us: it explains how rational people can formulate hypotheses which, in fact, truly describe parts of the mathematical realm – they can do this by merely constructing consistent purely mathematical theories. Of course, one might still wonder about our ability to *know* things (in the skeptic's sense) about the mathematical realm; but this is irrelevant in the present context, for all I am trying to establish is that my FBP-based epistemology does everything in the mathematical case that the perception-based epistemology does in the empirical case (and the latter does no better against skepticism than the former does).

I began this section with the worry that FBP *only* explains how our mathematical beliefs could turn out to be, in fact, reliable. The response, in a nutshell, is that this is exactly what *needs* to be explained, because the whole force of Benacerraf's argument lies in the fact that it makes us wonder how, if platonism were true, our mathematical beliefs could even be, in fact, reliable.

The only remaining hole in my argument is (1). I will motivate this premise in Section 5. But before I do that, I would like to provide a (partial) defense of FBP. Now, we have already seen that I do not *need* to argue for FBP; the reason is that I am simply not trying to establish, in this paper, that FBP is true; I am only trying to establish that there is no epistemological problem with FBP (and, moreover, I am only trying to construct an *externalist* epistemology for FBP). But I think it wise to say a few words in this connection in order to block the objection that I have only solved the epistemological problem with platonism by adopting an untenable version of platonism.

4. FBP AND THE NOTION OF MATHEMATICAL TRUTH

I will begin by fending off various objections to FBP; then at the end of this section, I will argue briefly that FBP is actually the best version of platonism there is, i.e., that non-full-blooded versions of platonism are untenable.

The main objection to FBP concerns the notion of mathematical truth. In particular, one might worry that if we adhere to a correspondence theory of truth, then FBP entails the claim that – among purely mathematical theories – consistency is sufficient for truth. The problem with this is not just that it seems to represent a departure from the way in which we think about mathematical truth, but that it seems to lead to a contradiction! For there are numerous cases where a mathematical sentence and its negation are both consistent. For instance, since ZFC and ZF+not-C (i.e., Zermelo-Fraenkel set theory with and without the axiom of choice, C) are both consistent (assuming that ZF is consistent) it follows from FBP, together with a correspondence theory of truth, that they are both true. Thus, it follows that C and not-C are both true.

But this is not a genuine contradiction. According to FBP, both ZFC and ZF+not-C truly describe parts of the mathematical realm; but there is nothing wrong with this, because they describe *different* parts of that realm. This might be expressed by saying that ZFC describes the universe of $sets_1$, while ZF+not-C describes the universe of $sets_2$, where $sets_1$ and $sets_2$ are different *kinds* of things.[21] (This, of course, oversimplifies matters, for there is more than one kind of object described by ZFC; that is, there is more than one universe in which ZFC is true. There are, for instance, universes in which ZFC and the continuum hypothesis (CH) are true and others in which ZFC and not-CH are true.) Thus, while we *can* derive the truth of both C and not-C, we can *only* do this by interpreting C in two different ways in the two different cases. Therefore, insofar as 'C and not-C' is true, it is no more a genuine contradiction than is the sentence 'Aristotle married Jackie Kennedy and Aristotle did not marry Jackie Kennedy'. (And note that since, in mathematics, we never allow a term to shift meaning within a theory, 'C and not-C' will not be a theorem of any of our mathematical theories, except those that contain an unrelated contradiction.[22])

It is worth noting that FBP does *not* advocate a *shift* in our conception of mathematical truth. Now, it *does* imply (when coupled with a correspondence theory of truth) that the consistency of a mathematical sentence is sufficient for its truth. But this is not because it has changed the meaning of 'true'; it's saying that (in purely mathematical contexts) consistency is sufficient for the *ordinary* notion of truth. What mathematicians *ordinarily* mean when they say that some set-theoretic claim is true is that it is true of the *actual* universe of sets. Now, as we have seen, according to FBP, there is no *one* universe of sets. There are many. But, nonetheless, a set-theoretic claim is true just in case it is true of *actual* sets. What FBP says is that there are so many different kinds of sets that every consistent set theory is true of an *actual* universe of sets.

Now, mathematicians do not just speak of *truth simpliciter*. They also speak of *truth in M*, where *M* is a precisely specified model. (Indeed, it is this notion that Tarski precisely defined.) But FBP also accords quite well with this aspect of mathematical practice. For according to FBP, before we can answer a question about, say, the truth of CH, we have to specify what sort of sets we are talking about; that is, we have to specify a domain of discourse. If we are speaking of $sets_{17}$ (i.e., one of the kinds of objects that is described by ZF+CH) then CH is true; if we are speaking of $sets_{12}$ (which are described by ZF+not-CH) then it is false; if we are speaking of ZF, then there is no straightforward answer, because that theory describes different sorts of objects (e.g., both $sets_{12}$ and $sets_{17}$) and CH holds for some of them and fails for others. But to specify what we are talking about in this way is essentially to specify a model; for a model is just an interpretation.

(Talk of *truth in M* gives rise to a second way in which mathematicians think of *truth simpliciter*, viz., as *truth in the standard model*. There is no conflict between this and FBP; we need only note that, according to FBP, there is no *metaphysical* distinction between standard and non-standard models; which models get counted as standard depends upon facts about *us*. This, of course, is not to say that there can be no good reason for singling out one (sort of) model as standard;[23] it simply means that such models do not enjoy a privileged ontological status. It is also worth noting that appeal to standard models can be *useful* to advocates of FBP in fending off certain objections. For instance, one might complain that FBP entails that '2 + 2 = 5' is true of some part of the mathematical realm, when we know that it is false. But according to FBP-ists, what we mean when we say that this sentence is false is that it is false in the standard model of arithmetic.[24])

A second objection that one might raise against FBP is that it seems to forbid us – for no good reason – from speaking of *all* sets. That is, even if the mathematical realm is as robust as FBP suggests, we ought to be able to develop a theory of *all* sets and say whether CH is true in this theory. My response to this is that we *can* develop such a theory. Tell me what you mean by 'set', and I will give you a theory of *all* the objects falling under that concept. Now, if you say, "I mean 'set' in the *broadest* sense", I will not be able to give you a theory, because you will not have given me a precise mathematical notion to work with. But as soon as you specify *precisely* what is meant by "the broadest sense of 'set' ", we will (assuming our definition is consistent) be able to construct a mathematical theory.[25]

(More generally, one might object that FBP prohibits us from quantifying over the *entire mathematical realm*. But this is just wrong; it may be that there is nothing interesting to say about the entire mathematical realm, but FBP certainly allows us to say things like 'All mathematical objects are mathematical objects'. Now, I suppose that one might worry that a sentence like 'There is no number 7' is consistent, but – if taken as about the entire mathematical realm – false. But this is just *not* a problem for FBP. FBP tells us that every consistent purely mathematical sentence truly describes part of the mathematical realm; the above sentence is not a counterexample to this claim, because it *does* truly describe part of the mathematical realm. Now, of course, this sentence is *not* true of the *entire* mathematical realm, but that's irrelevant, because FBP does *not* claim that every sentence that's true of some part of the mathematical realm is true of all of it.)

A third possible objection one might raise is that FBP seems to sacrifice the objectivity of mathematics. But this is just false. According to FBP, mathematical theories are true of an objective mathematical realm; that is, they are true independently of us. Now, one might respond here that, in spite of the fact that FBP-ists can countenance the existence of mathematical sentences which are objectively true, they cannot countenance the existence of mathematical disputes with any objective bite. For so long as the parties on either side of a purely mathematical debate avoid contradiction, FBP dictates that *both* parties are correct. But, again, this objection is misguided. There are at least two ways in which FBP-ists can salvage the objective bite of mathematical disputes. The first has to do with the notion of *inclusiveness*, or *broadness*: the dispute over CH, for instance, might be construed as a dispute about whether ZF+CH or ZF+not-CH characterizes a broader notion of 'set'. And a second way in which FBP-ists can salvage objective bite is by pointing out that certain mathematical disputes are disputes about whether some sentence is true in a *standard model*.

In addition to fending off the objections to FBP, I would also like to argue in its favor; that is, I would like to argue that it is the best version of platonism there is. The most important advantage that FBP has over non-full-blooded versions of platonism (i.e., versions of platonism in which there is just *one* kind of set and in which CH is either true or false) is that all of the latter fall prey to Benacerraf's epistemological argument. If non-full-blooded platonism were correct, it would be a mystery how we could ever know whether CH was true or false; or, in the lingo of FBP, it would be a mystery how we could know whether *the* universe of sets was a universe of $sets_1$, or $sets_2$, or . . . .[26]

But there are also *independent* reasons for favoring FBP over other sorts of platonism. For instance, FBP reconciles the objectivity of mathematics (which all platonists are committed to) with the legitimacy in mathematics of pragmatic modes of justification.[27] It is often said that the adoption of a new axiom for a mathematical theory can be justified pragmatically, e.g., because it answers certain open questions. FBP easily explains this. Consider, for instance, the question of whether to adopt CH as an axiom in set theory. Since there exist sets for which CH holds *and* sets for which it fails, it is legitimate to study either. The decision to 'adopt' CH is just the decision to study a certain kind of set; thus, there is nothing wrong with motivating this decision pragmatically. Non-full-blooded platonists, on the other hand, cannot account for this. Since, according to their view, CH is either true or false of the *entire* domain of sets, it is unclear why pragmatic modes of justification should be legitimate. Why should the fruitfulness of a claim about *the* universe of sets have anything to do with its truth?[28]

A second (related) advantage of FBP is that it reconciles the objectivity of mathematics with the extreme *freedom* that mathematicians have. As I have already pointed out, mathematicians cannot be criticized on the grounds that the objects of their (consistent and pure) theories do not exist. Indeed, just the opposite seems to obtain: one way for a mathematician to become famous is to develop an interesting theory about a kind of mathematical entity or structure of which no one has yet conceived. (Now, of course, a physicist could also become famous in this way, but again, before we would accept the new physical theory, we would demand independent evidence that the objects in question exist.)

It is worth noting that Field (if not Benacerraf) has, in so many words, admitted both that my treatment of the above alleged contradiction is acceptable and that FBP is the most natural version of platonism there is.[29] In considering two platonists, one of whom advocates GCH and one of whom denies it, he argues that there is no reason for either of them to consider the other as *wrong*, that they should, instead, simply say that one "has a less inclusive notion of set" than the other. And a bit later he implies that platonists are *committed* to the claim that "any consistent [purely] mathematical theory comes out true on an interpretation intended by its advocates."

5. CONSISTENCY

It remains only to justify premise (1) of my argument against Benacerraf. To this end, I need to argue that FBP-ists can account for the fact that human beings can – without coming into contact with the mathematical realm – know of certain purely mathematical theories that they are consistent. I have two arguments for this claim. The first is that platonists can give any account of our knowledge of mathematical consistency that anti-platonists can. In other words, if anti-platonists can account for knowledge of consistency, then platonists can too, because they can simply *borrow* the anti-platonist account. Now, of course, this is not *generally* true; platonists cannot *always* use anti-platonist accounts of mathematical knowledge, because the two groups of philosophers have different conceptions of what mathematical knowledge *is*; platonists think that such knowledge is about mathematical objects, and anti-platonists do not. But this situation changes in the special case of our knowledge of mathematical *consistency*: there is nothing to stop platonists from understanding consistency precisely as anti-platonists do. Thus, there is nothing to stop them from accounting for knowledge of consistency precisely as anti-platonists do.

What if it turns out that there is *no* acceptable anti-platonist account of consistency? Well, then this whole discussion will be moot; for, since we clearly cannot dispense with talk of consistency, we would have a knockdown argument for platonism. But I think there *are* ways for anti-platonists to talk about consistency. One way – developed by Kreisel and advocated by Field[30] – is to take 'consistent' as a primitive term which is governed by two rules of use. In platonistic terms, these rules are (a) if a sentence is *semantically consistent* (i.e., if it has a model) then it is consistent; and (b) if a sentence is consistent, then it is *syntactically consistent* (i.e., it cannot be refuted in a system of formal logic). According to Field and Kreisel, this view of consistency yields a very satisfying understanding of the completeness theorem: in proving that (among first-order theories) syntactic consistency implies semantic consistency, we prove that (among first-order theories) the intuitive notion of consistency is *de facto* coextensive with both technical notions of consistency. Now, it seems clear that platonists can adopt this view of consistency as easily as anti-platonists can; in fact, in trying to support this view, Field argues (on independent grounds) that platonists not only *can* adopt this view, but, by their own lights, *should* adopt it.

Now, one might claim that anti-platonists do not have to account for knowledge of mathematical consistency, because they need not admit that we *have* such knowledge. But anti-platonists *cannot* deny that we have such knowledge, because (a) it is entirely obvious that we *do* have such knowledge, and (b) the Benacerrafian argument *assumes* that we have such knowledge – the claim is that platonists cannot account for the fact that we do have mathematical knowledge. (In response to (b), one *could* claim that while we do have mathematical knowledge, we do not have any knowledge of mathematical consistency, i.e., we do not know that *any* of our mathematical theories or sentences – even those of the form 'Fa' – are consistent. But this is a wildly implausible view that no actual anti-platonist would ever hold; indeed, it's hard to even *imagine* someone so ignorant of mathematical consistency possessing mathematical knowledge.)

Another objection that anti-platonists might offer, I suppose, is that knowledge of consistency is *logical* knowledge, rather than mathematical knowledge. My response is that this is not an objection to my view. Knowledge of consistency *is* logical knowledge, and this explains why it is not problematic.[31] The whole point of this paper is that if we adopt FBP, then mathematical knowledge can arise directly out of logical knowledge. Platonists who do not accept FBP cannot make this claim, because they have to account for how someone could know which of our consistent mathematical theories are true and which are not, and this could not be logical knowledge. But FBP-ists do not have to account for this sort of knowledge, because according to them, *all* of our consistent purely mathematical theories are true of part of the mathematical realm.

All of this is, of course, reminiscent of Field's view; he argues that anti-platonists can take mathematical knowledge to be logical knowledge.[32] If I am right, then FBP-ists can do the same. And this does not commit them to logicism any more than Field's view commits *him* to logicism. Mathematical truth is not logical truth, because the existence claims of mathematics are not logically true.[33]

The second reason for thinking that FBP-ists can account for knowledge of mathematical consistency is, I think, more fundamental than the first. Indeed, it *explains* the first. The second reason is just this: in general – whether we are speaking of mathematical or physical objects – we do not need any access to (i.e., contact with) a set of objects in order to know whether a set of sentences about these objects is consistent. The reason this is so important is that the whole force of the epistemological argument against platonism derives from the fact that, at least *prima facie*, it appears that – in general – one *does* need contact with an object in order to attain knowledge of it; this is why the lack of contact with mathematical objects was a source of alarm. But if we are only concerned with knowledge of *consistency*, there is no need to be alarmed; for knowledge of the consistency of a set of sentences does *not* depend upon having access to the objects that the sentences are about, and so platonists will have no more difficulty than anti-platonists in accounting for our knowledge of mathematical consistency.

I can support my claim that knowledge of consistency does not depend upon access by merely considering examples. I do not need access to the

seventh child born in 1991 in order to know that the sentences asserting it to be female and Italian are consistent with each other; likewise, I do not need access to this child to know that the sentences asserting it to be male and not male are *in*consistent. The same is true of mathematical sentences. I do not need any access to the number 4 in order to know that '4 is even' and '4 is positive' are consistent, or that '4 is odd' and '4 is not odd' are inconsistent.

Now, I suppose that one might press me here to actually explain how we arrive at knowledge of mathematical consistency. But there are two reasons – both of which have already been given – why I needn't address this issue. First, this is just as problematic for anti-platonists as it is for platonists, and (as I've already pointed out) platonists can say whatever anti-platonists say here. Second, this is just as problematic for our knowledge of the consistency of empirical theories; thus, platonists do not have to address this issue in order to respond to Benacerraf, for (as I've already pointed out) Benacerraf's argument cannot work equally well against physical and mathematical objects. His argument can only work if it shows that there is a *special* epistemological problem with platonism that arises as a result of the inaccessibility of mathematical objects.

NOTES

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1 Some platonists define the view differently; see, e.g., Maddy (1990, p. 48).

2 Benacerraf (1973).

3 One might think that platonists could simply appeal to *proof* here. But this would be circular, for proofs rely upon prior knowledge of mathematical objects.

4 Field (1989, pp. 26–7).

5 Gödel (1964) thought that the mind is capable of forging some sort of contact with abstract objects. Maddy (1990, chapter 2) also adopts a contact-based epistemology, but for her, mathematical objects exist within spacetime; thus, she thinks they can actually be *seen*. In my (1994), I argue that Maddy's view is untenable; and in Chapter II of my (in progress), I argue that Gödel's is untenable.

6 My explanation is original, but it is related (in ways which I will make clear below) to some remarks of Michael Resnik (1982, p. 101). It is also worth noting that my epistemology bears some relation, although a much more distant one, to that of Jerrold Katz (1981, chapter VI) and David Lewis (1986, section 2.4). Their position is that since mathematical truths are *necessarily* true, we simply do not need any contact with the objects of these

truths in order to attain knowledge of them. On the surface, my view will appear utterly different, but I think it is related in certain relevant ways. I explain this in Chapter II of my (in progress); I also argue there that my epistemology is superior to that of Katz and Lewis.

[7] If we want to express FBP formally, it is done most intuitively in second-order logic. Letting 'Mx' mean 'x is a mathematical object' and letting x be a first-order variable and Y be a second-order variable, we can express FBP as follows: $(\exists x)(Mx)$ & $(Y)[\Diamond(\exists x)(Mx$ & $Yx) \supset (\exists x)(Mx$ & $Yx)]$. Shortly, we will see that the $\Diamond$ here is to be read as *logical* possibility; thus, since the existence of mathematical objects is logically possible, we can get rid of the existential clause of the definition, because the main clause will already entail that all possible mathematical objects exist.

[8] One might object that our mathematical theories do not describe *unique* parts of the mathematical realm, i.e., that all mathematical theories (even categorical ones) have multiple models. But I just don't see that this is a problem. If I know that some theory truly describes part of the mathematical realm, then I have knowledge of that realm; uniqueness is simply irrelevant. A related worry is that a theory might characterize a collection of objects that are different from the objects that its author *intended* to characterize. But again, this seems irrelevant: regardless of what the author intended, if a mathematical theory truly describes part of the mathematical realm, then one can attain knowledge of the mathematical realm by studying that theory. Also, one might question whether we 'intend' in mathematical contexts as we do in empirical contexts.

[9] It is not the structuralism in Resnik's view that is related to FBP.

[10] Resnik (1982, p. 101).

[11] Resnik makes a few mistakes in this connection; for instance, he thinks he must show that our mathematical theories *are* consistent and that we *can* know this. Resnik admits that this is a futile task. I will argue below, however, that it is also irrelevant; what needs to be argued is that *if* we have such knowledge, *then* platonists can adequately account for it. A second problem with Resnik's epistemology is that he doesn't acknowledge that he is relying upon FBP, much less *argue* that this is *acceptable*. Sections 3 and 4 of this paper will provide such an argument.

[12] Although Hilbert would never have formulated it as such, his view is also related to FBP. He once wrote in a letter to Frege that "if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence." See Frege (1980, pp. 39–40).

[13] In order to give a *complete* platonist epistemology, one would have to account for our knowledge of not just pure mathematical sentences, but impure mathematical sentences and mixed physical sentences as well. (Both mixed and impure sentences refer to physical *and* mathematical objects.) But we need not give a complete epistemology in order to refute Benacerraf's argument; we need only explain how human beings can attain *some* knowledge of the mathematical realm. Thus, I need only concern myself with pure mathematics. Nonetheless, my epistemology *can* be generalized to cover impure and mixed sentences by merely shifting from talk of consistent theories to talk of theories which do not imply any falsehoods about the physical world. I thank Hartry Field for pointing this out to me. (Note that this shift is irrelevant in the pure case, because here, the theories which imply no falsehoods about the physical world just *are* the consistent theories. This is because inconsistent pure theories imply *everything* about the physical world, whereas consistent pure theories imply *nothing* about the physical world, i.e., consistent pure theories are *conservative*, in Field's (1989, p. 58) sense of the term.)

[14] Note the scope of the existential quantifier here. I am not merely saying that FBP-ists can account for our knowledge that there exist consistent mathematical theories. Rather, I am saying that there exist *particular* mathematical theories such that FBP-ists can account for the fact that we can know that *these theories* are consistent.

[15] Note, too, that I am not claiming that we are led to accept a theory T by the mere belief that it is consistent. Mathematicians might have other criteria that need to be satisfied before they accept a theory. I am merely saying that the consistency of T is *one* of the factors, and that if T was believed to be inconsistent, we would *not* accept it.

[16] We could make (b) a bit less general, so that it reads: in constructing an epistemology for a theory T, it is legitimate to assume that T is true.

[17] The FBP-ist account of (M1) is simple: we can learn what FBP says and recognize that if FBP is true, then *any* theory like T (i.e., any consistent purely mathematical theory) truly describes part of the mathematical realm.

[18] Katz has made a similar point. He says (1981, p. 212) that "Empirical knowledge . . . has no advantage over *a priori* knowledge in encounters with the skeptic."

[19] Even if we were trying to provide an internalist account, it still would not matter whether any actual knowers accept FBP (at any level). For insofar as Benacerraf's claim is that human beings *could not* acquire knowledge of a mathematical realm, all that is needed to refute his claim is an account of how human beings *could* acquire such knowledge. (Actually, it is a bit more complicated: platonists need an account for which it is *plausible* that an *actual* human being *could* acquire knowledge in the specified way. If it were only necessary to say how human beings *logically could* acquire mathematical knowledge, the project would be easily accomplished: one could simply claim that there *could be* a pre-established harmony between our belief states and the mathematical realm. Obviously, this would not do: we need an account that is plausible in the above way.)

[20] One such method tells us to believe *all* consistent purely mathematical theories. Others demand that a theory satisfy other conditions, in addition to consistency. But if FBP is true, then all of these methods will be reliable.

[21] To say that $sets_1$ and $sets_2$ are different *kinds* of things might be slightly misleading, for we might want to say that different kinds of sets are generated by relativizing our quantifiers; thus, on this way of looking at things, a single set will be able to qualify as both a set_1 *and* a set_2. But for the present purposes, this is unimportant.

[22] This last point is important, for it enables us to block the worry that 'C and not-C' is *true but not satisfiable*. If we are working within a formal mathematical theory, 'C' and 'not-C' will contradict one another, and so 'C and not-C' will be false. The only way that 'C and not-C' can be true is if we are in an informal extra-mathematical setting in which the two occurrences of 'C' are read differently. But in this *informal* setting, there is nothing wrong with saying that 'C and not-C' is informally satisfiable, i.e., that it can be given an informal interpretation that makes it true, viz., one that interprets the two occurrences of 'C' in different ways. Since we are only speaking of interpretation and satisfiability *informally* here, there is nothing wrong with this.

[23] Reasons for taking some model as standard could stem (for instance) from the inclusiveness of its universe, or from its similarity to what was intended or to our pre-theoretic intuitions.

[24] One might think that this gives rise to an epistemic problem of how we can know what the standard model is like. But this isn't a problem for the very reason that standard models aren't metaphysically special; since what gets counted as standard is, in some sense, dependent upon us, it is not beyond our epistemic reach.

[25] Another solution to this problem is to point out that we could (if we wanted to) develop an *amalgamated* theory with a different kind of variable for each different kind of set that we want to talk about.

[26]In my (in progress) I have provided the argument for this claim, i.e., I have shown that other platonist epistemologies – e.g., those of Gödel, Maddy, and Quine – fail.
[27]See Maddy (1990, chapter 4) for a discussion of this problem.
[28]Non-full-blooded platonists might object that pragmatic considerations are relevant in empirical as well as mathematical theory construction. But the two cases are radically different. If an empirical hypothesis is pragmatically useful, we seek independent (non-pragmatic) confirmation for it, and until such confirmation is obtained, the hypothesis is considered suspicious and *ad hoc*; but this is not true in mathematics.
[29]Field (1989, pp. 276–78).
[30]Kreisel (1967) and Field (1991).
[31]Logical knowledge might not always be unproblematic, but it is never more problematic for platonists than it is for anti-platonists. Now, one might point out in this connection that Gödel has shown that knowledge of consistency is often equivalent to arithmetical knowledge. But first of all, Gödel only showed that knowledge of *syntactic* consistency is equivalent to arithmetical knowledge; thus, we can simply ignore this point here, for (a) we are only concerned here with knowledge of anti-platonistic kinds of consistency (e.g., the Kreisel/Field kind) but (b) syntactic consistency is a platonistic notion. And second of all, platonists can simply ignore complex cases of mathematical consistency; for in order to refute Benacerraf's argument, they need only explain how people can attain *some* mathematical knowledge; thus, they can concentrate on very simple cases.
[32]Field (1989, essay 3).
[33]The point that FBP is not a version of logicism can be made more explicit. Let T be some purely mathematical theory implying the existence of certain mathematical objects. According to FBP-ists, knowledge of the logical fact that T is consistent is sufficient for knowledge of the mathematical fact that T truly describes part of the mathematical realm; but FBP-ists do *not* claim that 'T is consistent' is *equivalent* to 'T truly describes part of the mathematical realm', because whereas the latter can only be true if there are mathematical objects, the former can be true even if there are no mathematical objects (assuming that anti-platonists have an account of consistency). What we can say, however, is that 'T truly describes part of the mathematical realm' follows from the conjunction of FBP and 'T is consistent'.

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