ALGEBRA COMPREHENSIVE EXAMINATION

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Directions: Answer 5 questions only. You must *answer at least one* from each of linear algebra, groups, and synthesis. Indicate CLEARLY which problems you want us to grade. Otherwise, we will select which ones to grade, and they may not be the ones that you want us to grade. Be sure to show enough work that your answers are adequately supported.

Notation: As usual, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{C}$, and $\mathbb{R}$ denote the sets of natural numbers, rational numbers, integers, integers modulo n, complex numbers, and real numbers respectively, regarded as groups or fields or vector spaces in the usual way.

Linear Algebra

(L1) Let V be a vector space. Suppose $u, v, w \in V$ satisfy $2u + v = w$ and $u + 2v = w$. What does this tell you about the dimension of the subspace spanned by $\{u, v, w\}$?

(L2) Let V and W be vector spaces, and let $T\colon V \to W$ be a linear map. Let $v_1, \ldots, v_k \in V$.

- (a) Assume $v_1, \ldots, v_k$ are linearly independent. Does it follow that $T(v_1), \ldots, T(v_k)$ are linearly independent? If yes, prove it. If no, find a condition on T that would allow us to conclude that $T(v_1), \ldots, T(v_k)$ are linearly independent, and prove it.
- (b) Assume $T(v_1), \ldots, T(v_k)$ are linearly independent. Does it follow that $v_1, \ldots, v_k$ are linearly independent? If yes, prove it. If no, find a condition on T that would allow us to conclude that $v_1, \ldots, v_k$ are linearly independent, and prove it.

(L3) Let V be an n-dimensional vector space, and let $T\colon V \to V$ be an invertible linear operator. Let $v \in V$ be an eigenvector of T with eigenvalue λ.

- (a) Prove that $\lambda \neq 0$.
- (b) Prove that v is also an eigenvector of T^{-1}. What is the eigenvalue of T^{-1} corresponding to v?

Groups

(G1) Let G be a group. Show that the following conditions are equivalent:

- (a) G has exactly two subgroups.
- (b) $G \cong \mathbb{Z}_p$ for some prime number p.

(G2) Let H be a subgroup of S_n for some $n \geq 2$. Show that, either all elements of H are even, or exactly half are even and half are odd.

(G3) Consider a group G acting on a non-empty set X. We say the group action is **transitive** if, for every $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$. Given $g \in G$ and $x \in X$, x is a **fixed point** of g if $g \cdot x = x$. We say that the group action is **free** if the identity element of G is the only group element with a fixed point (that is, if $g \cdot x = x$ for some x, then g is the identity element of G). Prove that, if the action of G on X is both transitive and free, then $|G| = |X|$.

Synthesis

(S1) Let V be a vector space over a field F. Let $G = \{T : V \to V \mid T \text{ is a linear transformation}\}$.

(a) Show that G is a group under function addition. That is, the group operation is defined to be $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ when $T_1, T_2 \in G$.

(b) Let

$$H = \{T \in G \mid \text{ there exists some } \alpha \in F \text{ such that } T(x) = \alpha x \text{ for all } x \in V\}$$

Show that H is a subgroup of G.

(S2) Let $SO_2(\mathbb{R})$ be the group of 2×2 orthonormal matrices (that is, $A^{-1} = A^T$) with determinant 1 and real entries. Let $U(1) = \{z \in \mathbb{C} : |z| = 1\}$, considered as a group under multiplication. Prove that $SO_2(\mathbb{R})$ and $U(1)$ are isomorphic.

(S3) Let $GL_2(\mathbb{R})$ be the group of 2×2 invertible matrices over the real numbers, and $\mathbb{R}^*$, the group of nonzero real numbers under multiplication. Since the determinant function $\det : GL_2(\mathbb{R}) \to \mathbb{R}^*$ satisfies $\det(AB) = (\det A)(\det B)$ for all $A, B \in GL_2(\mathbb{R})$, it is a group homomorphism. Let G be a finite subgroup of $GL_2(\mathbb{R})$.

(a) Show that $\det(G) \leq \{1, -1\}$.

(b) Show that either, all matrices in G have determinant 1, or exactly half of the matrices in G have determinant 1.