ALGEBRA COMPREHENSIVE EXAMINATION

Fall 2022

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Directions: *Answer 5 questions only.* You must answer *at least one* from each of linear algebra, groups, synthesis. Indicate CLEARLY which problems you want us to grade—otherwise, we will select which ones to grade, and they may not be the ones that you want us to grade. Be sure to show enough work that your answers are adequately supported.

Notation: $\mathbb{R}$ is the set of real numbers; $\mathbb{Z}_n$ is the set of integers modulo n. $GL(n,\mathbb{R})$ is the group of all invertible $n \times n$ matrices with real entries under matrix multiplication.

Linear Algebra

(L1) Let $v_1, v_2, v_3, v_4, w_1, w_2, w_3$ be vectors in a vector space. Suppose that

$$\{v_1, v_2, v_3, v_4\} \subseteq \operatorname{span}[\{w_1, w_2, w_3\}].$$

Explain very carefully why $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. Write down the statements of the theorems that you are using.

(L2) Let V be a vector space and $P : V \to V$ a linear map such that $P(P(v)) = P(v)$ for all $v \in V$. (P is called a **projection operator**.) Show that every $v \in V$ can be written uniquely as $v = u + w$ with $u \in \operatorname{im} P$ and $w \in \ker P$.

(L3) Let $\mathbb{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ be the set of all polynomials of degree 2 or less and let $\mathbb{M}_{2\times2}$ be the set of 2×2 matrices over $\mathbb{R}$. Then $\mathbb{P}_2$ and $\mathbb{M}_{2\times2}$ are vector spaces over $\mathbb{R}$ in the usual way. Let $h : \mathbb{M}_{2\times2} \to \mathbb{P}$ be defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (b + c)x + (a + c)x^2$$

(a) Prove that h is a linear transformation. Find the following for h: (b) range space, (c) null space, (d) rank

Groups

(G1) Show that every abelian group of order 60 contains an element of order 30.

(G2) Find all subgroups of the symmetric group S_4 that are isomorphic to the Klein Group, $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(G3) Let $\phi : G \to G'$ be a group epimorphism (i.e., a surjective homomorphism), and let $K = \ker(\phi)$, the kernel of ϕ. Let N be a subgroup of G that contains K. Show that N is a normal subgroup of G if and only if

$$\phi(N) = \{\phi(n) \mid n \in N\}$$

is a normal subgroup of G'.

Synthesis

(S1) Let $O(n,\mathbb{R})$ be the set of orthogonal $n \times n$ matrices over $\mathbb{R}$:

$$O(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid A^T = A^{-1}\}$$

where A^T is the transpose of A. Prove or disprove: $O(n,\mathbb{R})$ is a subgroup of $GL(n,\mathbb{R})$.

(S2) Let $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ and $H = \langle A \rangle$, the subgroup of $GL(2, \mathbb{R})$ generated by A. Show that H is not a normal subgroup of $GL(2, \mathbb{R})$.

(S3) Let V be a vector space and $GL(V)$ the group of all invertible linear transformations from V to V. (The group operation is composition, written $\circ$.) Let G be a subgroup of $GL(V)$. A subspace W of V is **invariant** with respect to G if, for all $w \in W$ and $g \in G$ we have $g(w) \in W$.

Let $\phi \in GL(V)$ be such that $g \circ \phi = \phi \circ g$ for all $g \in G$. Show that the subspaces $\operatorname{im} \phi$ and $\ker \phi$ are invariant with respect to G.