ALGEBRA COMPREHENSIVE EXAMINATION

Fall 2022

Brookfield*, Liu, Mijares

Directions: *Answer 5 questions only.* You must answer *at least one* from each of linear algebra, groups, synthesis. Indicate CLEARLY which problems you want us to grade—otherwise, we will select which ones to grade, and they may not be the ones that you want us to grade. Be sure to show enough work that your answers are adequately supported.

Notation: $\mathbb{R}$ is the set of real numbers; $\mathbb{Z}_n$ is the set of integers modulo n. $GL(n,\mathbb{R})$ is the group of all invertible $n \times n$ matrices with real entries under matrix multiplication.

Linear Algebra

(L1) Let $v_1, v_2, v_3, v_4, w_1, w_2, w_3$ be vectors in a vector space. Suppose that

$$\{v_1, v_2, v_3, v_4\} \subseteq \text{span}[\{w_1, w_2, w_3\}].$$

Explain very carefully why $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. Write down the statements of the theorems that you are using.

`Answer:` *Let $V = \text{span}[\{w_1, w_2, w_3\}]$.* ***Theorem A: Any spanning set contains a basis.*** *Hence V has a basis with 3 or less vectors.* ***Theorem B: The number of vectors in a basis is independent of the basis and is the dimension of the space.*** *Hence the dimension of V is three or less. By assumption $\{v_1, v_2, v_3, v_4\} \subseteq V$.* ***Theorem C: Any linearly independent set of vectors can be extended to a basis.*** *If $\{v_1, v_2, v_3, v_4\}$ is independent, then the dimension of V would be four or more, a contradiction. Thus $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.*

(L2) Let V be a vector space and $P : V \to V$ a linear map such that $P(P(v)) = P(v)$ for all $v \in V$. (P is called a **projection operator**.) Show that every $v \in V$ can be written uniquely as $v = u + w$ with $u \in \text{im}\, P$ and $w \in \ker P$.

`Answer:` *Let $v \in V$. Set $u = P(v)$ and $w = v - u$. Then $v = u + w$ and $u \in \text{im}\, P$ are clear. To check that $w \in \ker P$, we calculate*

$$P(w) = P(v - u) = P(v) - P(u) = P(v) - P(P(v)) = P(v) - P(v) = 0.$$

To show uniqueness, suppose that $v = u + w$ with $u \in \text{im}\, P$ and $w \in \ker P$. Then $u = P(u')$ for some $u' \in V$, $P(w) = 0$ and

$$P(v) = P(u + w) = P(u) + P(w) = P(P(u')) = P(u') = u$$

So u is uniquely determined by v and P, specifically $u = P(v)$. Then, of course, w is determined uniquely by $w = v - u$.

(L3) Let $\mathbb{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ be the set of all polynomials of degree 2 or less and let $\mathbb{M}_{2\times 2}$ be the set of 2×2 matrices over $\mathbb{R}$. Then $\mathbb{P}_2$ and $\mathbb{M}_{2\times 2}$ are vector spaces over $\mathbb{R}$ in the usual way. Let $h : \mathbb{M}_{2\times 2} \to \mathbb{P}$ be defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (b + c)x + (a + c)x^2$$

(a) Prove that h is a linear transformation. Find the following for h: (b) range space, (c) null space, (d) rank

`Answer:` *(a) Assume*

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \mapsto (b_1+c_1)x + (a_1+c_1)x^2,$$
$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \mapsto (b_2+c_2)x + (a_2+c_2)x^2.$$

Then

$$\begin{aligned}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} &= \begin{pmatrix} a_1+a_1 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix} \\ &\mapsto ((b_1+b_2)+(c_1+c_2))x + ((a_1+a_2)+(c_1+c_2))x^2 \\ &= (b_1+c_1)x + (a_1+c_1)x^2 + (b_2+c_2)x + (a_2+c_2)x^2\end{aligned}$$

Hence, h preserves vector addition.
For any $s \in \mathbb{R}$:

$$s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix} \mapsto (sb+sc)x + (sa+sc)x^2 = s((b+c)x + (a+c)x^2).$$

Hence h preserves scalar multiplication. Therefore, h is a linear transformation.
(b) The standard B basis for $\mathbb{M}_{2\times 2}$ is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The standard basis A for $\mathbb{P}_2$ is:

$$\{x^2, x, 1\}.$$

The matrix of h corresponding to the above two bases is:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the range space is by the first two rows as

$$Range(h) = \{rx + tx^2 : \; r, t \in \mathbb{R}\}.$$

and (c) the rank of h is 2. (d) The null space is determined by the last two columns (free variables):

$$\left\{ \begin{pmatrix} -c & -c \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} : c, d \in \mathbb{R} \right\}.$$

Groups

(G1) Show that every abelian group of order 60 contains an element of order 30.

`Answer:` *Let G be an abelian group of order 60. By the Classification Theorem of Finite Abelian Groups, either $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ or $G \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. Since $\mathbb{Z}_4$ contains a subgroup $\{0, 2\}$ isomorphic to $\mathbb{Z}_2$, either way, G contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. Since 2, 3 and 5 are primes, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ is cyclic, is isomorphic to $\mathbb{Z}_{30}$ and contains an element of order 30. Hence G contains an element of order 30.*

OR

Since 2, 3 and 5 are primes that divide $|G|$, by Cauchy's Theorem, G contains elements a, b, c of orders 2, 3 and 5 respectively. Let $H = \langle abc \rangle$ be the cyclic subgroup generated by abc. Since $(abc)^{30} = a^{30}b^{30}c^{30} = 1$, $|H|$ divides 30. Since $(abc)^{15} = a$, H contains an element of order 2 and 2 divides $|H|$. Since $(abc)^{10} = b$, H contains an element of order 3 and 3 divides $|H|$. Since $(abc)^6 = c$, H contains an element of order 5 and 5 divides $|H|$. Thus $|H| = 30$ and abc is an element of order 30.

(G2) Find all subgroups of the symmetric group S_4 that are isomorphic to the Klein Group, $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
`Answer:` *The Klein group contains three commuting elements of order two. S_4 has six 2-cycles of order two: $\{(1\ 2), (1\ 3), (1\ 4), (2\ 3), (2\ 4), (3\ 4)\}$ and three products of 2-cycles with order two: $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Pairs of disjoint 2-cycles commute so generate subgroups isomorphic to V_4, so*

$$\{1, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\} \quad \{1, (1\ 3), (2\ 4), (1\ 3)(2\ 4)\} \quad \{1, (1\ 4), (2\ 3), (1\ 4)(2\ 3)\}$$

are isomorphic to V_4. Not quite so obvious is that

$$\{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

is also isomorphic to V_4.

(G3) Let $\phi : G \to G'$ be a group epimorphism (i.e., a surjective homomorphism), and let $K = \ker(\phi)$, the kernel of ϕ. Let N be a subgroup of G that contains K. Show that N is a normal subgroup of G if and only if

$$\phi(N) = \{\phi(n) \mid n \in N\}$$

is a normal subgroup of G'.
`Answer:` *Suppose that N is normal in G. We show $\phi(N)$ is normal in G'. Let $g' \in G'$. Then, since ϕ is onto, $g' = \phi(g)$ for some $g \in G$. Since we also have $g'^{-1} = \phi(g^{-1})$, this gives*

$$g'\phi(N)g'^{-1} = \phi(g)\phi(N)\phi(g^{-1}) = \phi(gNg^{-1}) = \phi(N)$$

showing that $\phi(N)$ is normal in G'.

Now suppose that $\phi(N)$ is normal in G'. We show that N is normal in G. Let $g \in G$ and $n \in N$. Then $\phi(g) \in G'$, $\phi(n) \in \phi(N)$ and $\phi(g)\phi(n)\phi(g^{-1}) \in \phi(N)$. That is, $\phi(g)\phi(n)\phi(g^{-1}) = \phi(m)$ for some $m \in N$. Since $\phi(m)^{-1} = \phi(m^{-1})$ we have $\phi(gng^{-1}m^{-1}) = 1$, that is, $gng^{-1}m^{-1} \in \ker(\phi) = K$ and so $gng^{-1} \in mK$. Because $K \subseteq N$ and $m \in N$, we have $mK \subseteq N$ and then $gng^{-1} \in N$. This shows that N is normal in G.

Synthesis

(S1) Let $O(n, \mathbb{R})$ be the set of orthogonal $n \times n$ matrices over $\mathbb{R}$:

$$O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid A^T = A^{-1}\}$$

where A^T is the transpose of A. Prove or disprove: $O(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.
`Answer:` *Proof: Let A, B be in $O(n, \mathbb{R})$. Then $A^T = A^{-1}$ and $B^T = B^{-1}$, that is $A^TA = I$ and $B^TB = I$. Then using known properties of the transpose*

$$(AB)^T(AB) = B^TA^T(AB) = B^T(A^TA)B = B^TIB = B^TB = I$$

This shows that $(AB)^T = (AB)^{-1}$, that is, $AB \in O(n, \mathbb{R})$ and $O(n, \mathbb{R})$ is closed under the group operation.

If $A \in O(n, \mathbb{R})$, then $A^T = A^{-1}$ and, since the transpose and inverse operations commute,

$$(A^{-1})^T = (A^T)^{-1} = (A^{-1})^{-1}$$

so $A^{-1} \in O(n, \mathbb{R})$ and $O(n, \mathbb{R})$ is closed under taking inverses.

This suffices to show that $O(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.

(S2) Let $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ and $H = \langle A \rangle$, the subgroup of $GL(2, \mathbb{R})$ generated by A. Show that H is not a normal subgroup of $GL(2, \mathbb{R})$.

`Answer:` *By direct calculation A has order 3 and*

$$H = \{I, A, A^2\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\}$$

To show that H is not normal in $GL(2, \mathbb{R})$ it suffices to find a matrix $B \in GL(2, \mathbb{R})$ such that BAB^{-1} is not in H. Almost any matrix in $GL(2, \mathbb{R})$ would suffice. For example, if $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $BAB^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ is not in H.

(S3) Let V be a vector space and $GL(V)$ the group of all invertible linear transformations from V to V. (The group operation is composition, written $\circ$.) Let G be a subgroup of $GL(V)$. A subspace W of V is **invariant** with respect to G if, for all $w \in W$ and $g \in G$ we have $g(w) \in W$.

Let $\phi \in GL(V)$ be such that $g \circ \phi = \phi \circ g$ for all $g \in G$. Show that the subspaces $\operatorname{im} \phi$ and $\ker \phi$ are invariant with respect to G.

`Answer:`

(a) *If $v \in \operatorname{im} \phi$, then $v = \phi(w)$ for some $w \in V$. So, for all $g \in G$,*

$$g(v) = g(\phi(w) = \phi(g(w)) \in \operatorname{im} \phi$$

so $g(v) \in \operatorname{im} P$.

(b) *If $v \in \ker \phi$, then $\phi(v) = 0$. So, for all $g \in G$,*

$$\phi(g(v)) = g(\phi(v) = g(0) = 0$$

so $g(v) \in \ker P$.