ALGEBRA COMPREHENSIVE EXAMINATION
FALL 2000

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Answer 5 questions only. You must answer *at least one* from each of Groups, Rings, and Fields.
Please show work to support your answers.

GROUPS

1. Prove: (a) A group of order 80 need not be abelian.
   (b) A group of order 80 must be solvable.

2. Let H and K be subgroups of a group G with K normal in G and G = <H, K>. If H and K are each solvable, prove that G is solvable.

3. Let P be a Sylow p-subgroup of G and N a normal subgroup of G. Prove
   (a) $P \cap N$ is a Sylow p-subgroup of N .
   (b) PN/N is a Sylow p-subgroup of G/N.

RINGS

1. Let R and S be rings with identity, $1 \in R$, $S \neq 0$, and let $f: R \to S$ be a ring homomorphism of R onto S. Prove:
   (a) S has an identity and it is f(1) .
   (b) If R is commutative, then S is commutative.
   (c) If a is a unit element of R, then f(a) is a unit element of S.

2. Let R be a commutative ring with identity 1. A principal ideal is an ideal generated by a single element and R is a Principal Ideal Ring (PID) if every ideal is principal. Since the ring of integers **Z** has a Euclidean algorithm, it is PID and you may use that fact that if needed.
   (a) Find and verify a principal generator for the ideal generated by 8 and 12 in **Z**.
   (b) Prove that the homomorphic image of a PID is a PID
   (c) Prove that **Z**[x], the polynomials over **Z**, is not a PID.

3. Let R = $\{a/b \in \mathbf{Q} \mid b \text{ is odd}\}$ with the usual rational number operations.
   (a) Prove that R is an integral domain.
   (b) Find U(R), the group of units of R.
   (c) Prove that R\U(R) [= R - U(R)] is the unique maximal ideal in R.

FIELDS

1. Let E be the splitting field of $x^8 - 2$ over the rationals **Q**.
   (a) Prove that [E : **Q**] = 16.
   (b) Show that the Galois group G(E/**Q**) is not abelian.

2. Let $\mathbf{Z}_p = \{0, 1, 2, ..., p-1\}$ be the field of integers modulo p and f(x) an irreducible polynomial in $\mathbf{Z}_p[x]$ of degree n. Prove that f(x) is a factor of $x^{p^n} - x$.

3. Let F be the field of integers modulo 5 and let $f(x) = x^3 + 3x^2 + 3x + 2$. Prove that $f(x)$ is reducible over F, find the splitting field K, and determine the number of elements of K.