ALGEBRA COMPREHENSIVE EXAMINATION

Fall 2001

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Answer 5 questions only. You must answer at least one from each of Groups, Rings, and Fields. Please show work to support your answers.

Groups

1. Let G be a finite group. Set $C(x) = \{y \in G |\, xy = yx\}$ and define xRy iff $y = g^{-1}xg$ for some $g \in G$.

(a) Show $C(x)$ is a subgroup of G.

(b) Show R is an equivalence relation on G.

(c) Show $|x^G| = [G : C(x)]$ ($x^G = \{y \in G |\, yRx\}$, $[G : C(x)] =$ index of $C(x)$ in G).

2. Let G be a group of order $2^5 19^t$, t a positive integer. Prove that G is solvable.

3. Define $Z_1(G) = Z(G)$, $Z_n(G)/Z_{n-1}(G) = Z(G/Z_{n-1}(G))$. Prove that if G is a p-group then $Z_n(G) = G$ for some n.

Rings

4. Let R be a commutative ring with identity 1. Assume $1 = e + f$ and $ef = 0$. Define $\phi : R \to R$ by $\phi(x) = ex$. Prove:

(a) e is an idempotent (i.e. $e^2 = e$).

(b) ϕ is a ring homomorphism.

(c) e is the identity of $\phi(R)$ (the image of ϕ).

5. Let A be an ideal of the commutative ring R. Set $\rho(A) = \{x \in R |\, x^n \in A,\ \text{for some } n > 0\}$.

(a) Show that $\rho(A)$ is an ideal.

(b) Show that $\rho(\rho(A)) = \rho(A)$.

(c) $\rho(A \cap B) = \rho(A) \cap \rho(B)$.

6. Let R be a commutative ring with identity. Let Q be an ideal of R, and let P be the ideal $\{x \in R |\, x^n \in Q \text{ for some positive integer } n\}$. Definition: If A, B are ideals then $AB = \{\sum a_i b_i : \text{all finite sums}\}$. Prove that if P is maximal and if $Q = P_1 P_2 \neq R$ for some prime ideals P_1 and P_2, then $Q = P^2$.

Fields

7. Let E be the splitting field of $x^8 - 2$ over Q.

(a) Prove that $[E : Q] = 16$.

(b) Show that the Galois group $\mathcal{G}(E/Q)$ is not abelian.

8. Let E be the splitting field of $p(x) = x^3 + 11x + 3$ over Q.

(a) Prove that $p(x)$ is irreducible over Q.

(b) Find $a, b, c \in Q$ such that $(\theta^2 + 1)^{-1} = a + b\theta + c\theta^2$, for a root θ of $p(x)$.

9. Prove or disprove.

(a) $Q(\sqrt[3]{2})$ is a normal extension of Q.

(b) $Q[x]/(x^4 - 2)$ is a normal extension of Q.