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Bishop
Cates

ALGEBRA COMPREHENSIVE EXAMINATION
FALL 2002

Answer **FIVE** questions only. You must answer **AT LEAST ONE** from each of **GROUPS**, **RINGS**, and **FIELDS**. *Be sure to show enough work that your answers are adequately supported.*

GROUPS

1. Let p be a prime and G be a p–group. Let N be a normal subgroup of G and assume N has order p. Prove that N is in the center of G.

2. Let H be a proper subgroup of a finite group G and for each g in G let $H^g = \{g^{-1}h\,g \mid h \in H\}$. Let $K = \bigcup_{g\in G} H^g$. Prove that $K \neq G$.

3. Let G be a group of order $992 = 32\times 31$. Prove that G is solvable.

RINGS

1. Let R be a finite commutative ring with more than one element and with no zero divisors. Prove that R is a field.

2. Let R be a commutative ring with 1 such that all its ideals are finitely generated. Prove that any ascending chain of ideals

   $$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

   must terminate in finitely many steps.

3. Prove that every ideal of a Euclidean domain is principal.

FIELDS

1. Let E be the splitting field of $x^8 - 2$ over the field of the rational numbers $\mathbf{Q}$.
   i) Prove that $[E:\mathbf{Q}] = 16$.
   ii) Show that the Galois group $G(E/\mathbf{Q})$ is not abelian.

2. Let K be a field extension of a field F and $\alpha \in K$. Let $F[\alpha]$ be the smallest subring of K that contains both F and α and let $F(\alpha)$ be the smallest subfield of K that contains both F and α. Prove that α is algebraic over F if and only if $F[\alpha] = F(\alpha)$.

3. Let $\mathbf{Q}$ be the field of the rational numbers and let $F = \mathbf{Q}(\sqrt{3}, \sqrt{5}, \sqrt{11}, \sqrt{13})$. Find α in F such that $F = \mathbf{Q}(\alpha)$ and prove your result.