ALGEBRA COMPREHENSIVE EXAMINATION
Fall 2003

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Cates

Answer 5 questions only. You must answer *at least one* from each of Groups, Rings, and Fields.
Be sure to show enough work that your answers are adequately supported.

GROUPS

1. Let p be a prime and let G be a p-group. Let N be a normal subgroup of G of order p. Prove that N is in the center of G.

2. Prove that a group of order $3^2 \cdot 11^2$ must be solvable.

3. Let G be an abelian group of order pq, p and q distinct primes. Prove that G is cyclic.

RINGS

1. Let R and S be rings, R with unit element 1, and let ϕ:R→S be a ring homomorphism. Prove:
   (a) If ϕ is onto, then S has a unit element.
   (b) If R is commutative, S need not be commutative even if ϕ is 1-1 but, if ϕ is onto, S is commutative.
   (c) If S is a commutative ring with no zero divisors, then either $\phi(r) = 0$ for every $r \in R$ or $\phi(1)$ is the unit element of S.

2. Let D be a Euclidean domain. Prove:
   (a) If a divides bc and gcd(a, b) = 1 then a divides c.
   (b) If a is irreducible then a is prime.

3. Let R be a ring with identity. Ideals I and J are called **comaximal** if $I+J = R$. Let I_i, $i = 1,\dots, n$ be a collection of ideals that are pairwise comaximal; i.e., for $i \neq j$, I_i and I_j are comaximal. Prove that for any k, $1 \leq k \leq n$, the ideals I_k and $\bigcap_{i \neq k} I_i$ are comaximal.

FIELDS

1. Let F be the field of integers modulo 5 and let $f(x)= x^3+3x^2+3x+2$. Prove that f(x) is reducible over F, find the splitting field K and determine the number of elements of K.

2. Let **Q** be the field of rationals and let $p(x) = x^3 - 5x + 11$.
   (a) Prove that $p(x)$ is irreducible over **Q**
   (b) Let α is a root of $p(x)$.
      (i) Find a, b, c in such that $1/\alpha = a + b\alpha + c\alpha^2$.
      (ii) Find a, b, c in such that $1/(\alpha - 2) = a + b\alpha + c\alpha^2$.

3. Let F be a field.
   (a) Prove that every extension field of degree 2 is a normal extension.
   (b) Give an example of a field extension that is not normal (with confirmation, of course).