ALGEBRA COMPREHENSIVE EXAM
FALL 2005

Committee: Bishop, Brookfield*, Krebs

Answer 5 questions only. You must answer **at least one** question from each of **GROUPS, RINGS** and **FIELDS**. Please show work to support your answers.

GROUPS:

(1) Exhibit four distinct (i.e. nonisomorphic) groups of order 12, verifying that they are non-isomorphic.

(2) Let G be a group of order 242. Prove that G contains a nontrivial normal abelian subgroup H. (By nontrivial, we mean $H \neq \{e\}$.)

(3) Let H be a subgroup of a group G. Show that the following conditions are equivalent:
 (a) $x^{-1}y^{-1}xy \in H$ for all $x, y \in G$
 (b) H is a normal subgroup and G/H is abelian.

RINGS:

(1) Let $\mathbb{Z}_n[x]$ denote the ring of polynomials in x with coefficients in the ring of integers modulo n. Let $R = \mathbb{Z}_6[x]$. Let $I = (4) \subseteq R$. (In other words, I is the ideal in R generated by the constant 4.) Prove that:
 (a) The ring R/I is isomorphic to the ring $\mathbb{Z}_2[x]$, and
 (b) I is a prime ideal, and
 (c) I is not a maximal ideal.

(2) Suppose R is a commutative ring with identity, I is a proper ideal of R, and $a \in R$.
 (a) Prove that the smallest ideal of R which contains I and a is $I + (a)$ where

$$I + (a) = \{x \in R \mid x = i + ra \text{ for some } i \in I \text{ and } r \in R\}.$$

 (That is, show that (1) $I + (a)$ is an ideal, and (2), if J is any ideal which contains I and a, then J contains $I + (a)$.)
 (b) Prove that I is a maximal ideal if and only if it has the property that, if $a \in R$ and $a \notin I$, then $I + (a) = R$.

(3) Let R be the ring of matrices of form $\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}$ with $a, b \in \mathbb{Q}$. Prove that R is isomorphic to $\mathbb{Q}(\sqrt{2})$.

FIELDS:

(1) Explicitly produce an example of a field of 4 elements. Verify that it is a field and give its complete multiplication table. Hint: $f(x) = x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$.

(2) Let E be the splitting field of $(x^2 - 2)(x^2 - 3)$ over the field $\mathbb{Q}$ of rational numbers. Prove that the Galois group of E over $\mathbb{Q}$ is isomorphic to the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the group of integers mod 2.

(3) Let $F \subseteq E$ be an extension of fields.
 (a) Define what it means for E to be an algebraic extension of F.
 (b) If $(E : F) < \infty$, show that E is an algebraic extension of F.