**ALGEBRA COMPREHENSIVE EXAMINATION**

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Answer 5 questions only. You must answer *at least one* from each of groups, rings, and fields. Be sure to show enough work that your answers are adequately supported.

### Groups

1. Let $G$ be a group, and let $G'$ be its commutator subgroup. Let $\mathbb{Z}$ denote the group of integers under addition. Prove that if $G = G'$, then any homomorphism from $G$ to $\mathbb{Z}$ is the zero function.

2. Let $G$ be a group of order $175 (= 5^2 7)$. Prove that $G$ is abelian.

3. Let $p$ be a prime and assume $G$ is a finite $p$-group.

   (a) Show that the center of $G$ is non-trivial (i.e. $Z(G) \neq \{e\}$).

   (b) Let $N$ be a normal subgroup of $G$ of order $p$. Show that $N \subseteq Z(G)$.

### Rings

1. Prove that every ideal of a Euclidean domain is principal.

2. Let $R$ be a commutative ring with identity and $I$ be an ideal of $R$. Define

$$\sqrt{I} = \{x \in R \mid x^n \in I, \text{ for some } n \geq 1\}.$$

   Prove the following:

   (a) $\sqrt{I}$ is an ideal of $R$.

   (b) If $I \subseteq J$ are ideals, then $\sqrt{I} \subset \sqrt{J}$.

   (c) $\sqrt{\sqrt{I}} = \sqrt{I}$.

   (d) If $I$ and $J$ are ideals of $R$, then $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

3. Let $\mathbb{Z}[i]$ denote the ring of Gaussian integers. Let $\mathbb{Z}[x]$ denote the ring of polynomials with integer coefficients. Let $f$ be the unique ring homomorphism from $\mathbb{Z}[x]$ to $\mathbb{Z}[i]$ such that $f(1) = 1$ and $f(x) = i$.

   (a) Show that the kernel of $f$ is a prime ideal of $\mathbb{Z}[x]$.

   (b) Show that $\mathbb{Z}[x]/(x^2 + 1)$ is an integral domain.

### Fields

1. (a) Let $\mathbb{Z}_2$ denote the field with two elements. Let $F = \mathbb{Z}_2[x]/(x^2 + x + 1)$. Prove that $F$ is a field.

   (b) Let $R$ be the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$. Prove that the additive group of $F$ is isomorphic to the additive group of $R$.

   (c) Prove that $R$ is not isomorphic (as a ring) to $F$.

2. Let $E$ be the splitting field of $p(x) = x^6 - 2$ over the rationals $\mathbb{Q}$.

   (a) Find $[E : \mathbb{Q}]$ and explain.

   (b) Show that the Galois group $G(E/\mathbb{Q})$ is not abelian.

3. Let $K$, $L$, and $F$ be fields with $F \subseteq L \subseteq K$, $[L : F] = m$, and $[K : L] = n$. Prove that $[K : F] = mn$.