ALGEBRA COMPREHENSIVE EXAMINATION
SPRING 2000

Basmaji
Bishop*
Chabot

Answer 5 questions only. You must answer *at least one* from each of Groups, Rings, and Fields.
Please show work to support your answers.

GROUPS

1. Let G be a group, A(G) be the set of all automorphisms of G, and I(G) be the set of all inner automorphisms of G.
   Prove: (a) A(G) and I(G) are groups.
   (b) $I(G) \cong G / Z$ where $Z = Z(G)$ is the center of G.
   (c) I(G) is a normal subgroup of A(G).

2. Let H be a subgroup of a group G and assume $G = HZ$ where $Z = Z(G)$ is the center of G.
   Prove: (a) $H \cap Z = Z(H)$ where Z(H) is the center of H.
   (b) $G / Z \cong H / Z(H)$.

3. Let G be an group of order 175 ($= 5^2 7$). Prove that G is abelian.

RINGS

1. Let R be a commutative ring with identity. Assume $1 = e + f$ and $0 = ef$. Define $\phi(x) = ex$.
   Prove: (a) e is an idempotent of R [i.e., $e^2 = e$].
   (b) ϕ is a ring homomorphism.
   (c) e is the identity of $\phi(R)$ [the image of ϕ].

2. Let R be a commutative ring with identity 1 and let I be an ideal of R.
   Prove: (a) I is a maximal ideal iff R / I is a field.
   (b) I is a prime ideal iff R/I is an integral domain.
   (c) Every maximal ideal of R is prime.

3. A Principal Ideal Ring (PID) is a ring in which every ideal is principal. It is a fact and you may use the fact that since the ring of integers $\mathbf{Z}$ has a Euclidean algorithm, it is PID.
   Prove: (a) The homomorphic image of a PID is a PID
   (b) The integers modulo n, $\mathbf{Z}_n$ is a PID
   (c) $\mathbf{Z}_6[x]$, the polynomials over $\mathbf{Z}_6$, is not a PID.

FIELDS

1. Let E be the splitting field of $x^3 - 5$ over the rationals $\mathbf{Q}$. Find and describe every element of the corresponding Galois group $G(E/\mathbf{Q})$ and prove your result.

2. Let E be an algebraic extension of a field F. Let $\alpha \in E$ and set $p(x) = Irr(\alpha, x, F)$, the minimal polynomial of α over F.
   Prove: (a) If deg $p(x) = 5$, then $F(\alpha^2) = F(\alpha)$.
   (b) If $\beta \in E$ and $[F(\beta) : F] = 3$, prove that $p(x) = Irr(\alpha, x, F(\beta))$.

3. Let $E = \mathbf{Q}(\sqrt{3}, \sqrt{7})$ and $\alpha .= \sqrt{3} + \sqrt{7}$
   Prove: (a) $[E : \mathbf{Q}] = 4$.
   (b) $E = \mathbf{Q}(\alpha)$
   (c) Describe the Galois group $G(E/\mathbf{Q})$.