ALGEBRA COMPREHENSIVE EXAMINATION

Winter 2002

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Answer 5 questions only. You must answer at least one from each of Groups, Rings, and Fields. Please show work to support your answers.

GROUPS:

1. Let P be a Sylow p-subgroup of G. Let $N \lhd G$. Show:

(a) $P \cap N$ is a Sylow p-subgroup of N.

(b) PN/N is a Sylow p-subgroup of G/N.

2. Let H be a subgroup of G and let $Z = Z(G)$, the center of G, and suppose $G = HZ$. Prove:

(a) $H \cap Z = Z(H)$.

(b) $G/Z = H/Z(H)$.

3. Let G be a group of order 175 ($5^2 \cdot 7$). Prove that G is abelian.

RINGS:

4. (a) Let F be a field and let $f(x) \in F[x]$ with $\deg(f(x)) = n > 0$. Prove that $f(x)$ has at most n roots in F.

(b) Let F be a field and let $f(x)$ and $g(x)$ be elements of $F[x]$ with $\deg(f(x))$ and $\deg(g(x))$ each at most n. Suppose there exist $a_1, a_2, a_3, \ldots, a_{n+1} \in F$ such that $f(a_i) = g(a_i)$ for $1 \le i \le n+1$. Prove that $f(x) = g(x)$.

5. Prove that the ring $F^{2\times 2}$ of 2×2 matrices over the field F has no ideals except for $\{0\}$ and $F^{2\times 2}$.

6. Let M be a proper ideal of the commutative ring R. Prove that M is a maximal ideal if and only if $R = M + (a)$, for all $a \notin M$ (here $(a) =$ the principal ideal generated by a).

FIELDS:

7. Let E be an algebraic extension of a field F. Let $\alpha \in E$ and let $p(x) \in \mathrm{Irr}(\alpha, x, F)$, the minimal polynomial of α over F. Prove:

(a) If the degree of $p(x)$ is 3, then $F(\alpha^2) = F(a)$.

(b) If $\beta \in E$ and $[F(\beta) : F] = 7$, then $p(x) = \mathrm{Irr}(\alpha, x, F(\beta))$.

8. Let E be the splitting field of $x^5 - 3$ over the rational numbers Q.

(a) Find $[E : Q]$ and explain your answer.

(b) Show that the Galois group $\mathcal{G}(E/Q)$ is not abelian.

9. (a) Show that $f(x) = x^3 + 2x + 1$ is irreducible over the rational numbers Q.

(b) Show that $f(x)$ has at least one real root.

(c) Let α be a root of $f(x)$ in the reals and find rational numbers b_0, b_1, b_2 such that $(\alpha+1)^{-1} = b_0 + b_1\alpha + b_2\alpha^2$.