California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis Spring 2023

Da Silva*, Gutarts, Krebs

Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Spring 2023 #1. Use the definition of continuity to show that the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

is not continuous anywhere on $\mathbb{R}$.

Spring 2023 #2. Let a_n be the sequence

$$a_n = \begin{cases} 1 - \frac{1}{2^n} & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\liminf a_n = 0 \quad \text{and} \quad \limsup a_n = 1.$$

Spring 2023 #3. Let (a_n) be a sequence of real numbers.

a. State the definition of a Cauchy sequence.
b. Prove that if the sequence (a_n) converges, then it is a Cauchy sequence.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Spring 2023 #4. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $y, z \in \mathcal{H}$. Define $T\colon \mathcal{H} \to \mathcal{H}$ by

$$T(x) = \langle x, y \rangle z.$$

a. Prove that T is a linear transformation.
b. Prove that T is bounded.
c. Prove that T is continuous. Hint: Use parts (a) and (b).
d. Prove that $\|T\| \leq \|y\|\|z\|$. Here $\|T\|$ denotes the operator norm of T. Hint: Use your answer to part (b).

Spring 2023 #5. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be the function defined by setting

$$f(x) = \begin{cases} 1 + \frac{x}{\pi} & \text{if } -\pi \leq x < 0, \\ 1 - \frac{x}{\pi} & \text{if } 0 \leq x < \pi, \end{cases}$$

and then extending the result 2π-periodically.

a. Find the trigonometric Fourier series for $f(x)$.
b. Use the result from part (a) to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2},$$

and prove that your answer is correct.

Spring 2023 #6. Let H be a Hilbert space. Suppose that H is the orthogonal direct sum of two closed subspaces M and N. Moreover, suppose that E is an orthonormal basis for M, and suppose that F is an orthonormal basis for N. Prove that $E \cup F$ is an orthonormal basis for H.

Spring 2023 #7. Let I be the interval $[a, b]$ for some $a < b$, and let

$$C(I) = \{f : I \to \mathbb{R} : f \text{ is continuous}\}.$$

For $f, g \in C(I)$, define

$$d(f, g) = \sup_{x \in I} |f(x) - g(x)|.$$

Show that d defines a metric on $C(I)$.