Binary Strings Without Odd Runs of Zeros

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Abstract

We look at binary strings of length n which contain no odd run of zeros and express the total number of such strings, the number of zeros, the number of ones, the total number of runs, and the number of levels, rises and drops as functions of the Fibonacci and Lucas numbers and also give their generating functions. Furthermore, we look at the decimal value of the sum of all binary strings of length n without odd runs of zeros considered as base 2 representations of decimal numbers, which interestingly enough are congruent (mod 3) to either 0 or a particular Fibonacci number. We investigate the same questions for palindromic binary strings with no odd runs of zeros and obtain similar results, which generally have different forms for odd and even values of n.

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1 Introduction

Binary sequences are of great importance in computer science, where they encode instructions as well as decimal numbers using just the digits 0 and 1. Thus, most questions regarding binary sequences relate to their decimal values. However, one can also regard them as an "abstract" string of digits, very much like compositions. A composition of n is an ordered sequence of numbers whose sum is n, whereas a binary string of length n is an ordered sequence of n zeros and ones. Actually, there is a one-to-one correspondence between compositions of $n+1$ with odd summands and binary strings of length n without odd runs of zeros. The latter will be investigated in this article.

Alladi and Hoggatt [1] have studied compositions of n with summands 1 and 2, and found many connections to the Fibonacci sequence. Besides counting the number of such compositions, they looked at the number of occurrences of the individual summands and the number of levels (a summand followed by itself), rises (a summand followed by a larger summand) and drops (a summand followed by a smaller summand). Chinn et. al. [2, 3] have looked at these questions for compositions that allow all integers as summands, and Grimaldi has examined compositions without 1's [4] and compositions with odd summands [5], where he also looked at congruence questions.

In this paper we will explore similar questions for binary strings of length n without odd runs of zeros. Such a string is a sequence of n zeros and ones, where no odd number of zeros occur consecutively. A consecutive string (of maximal length) of either zeros or ones is called a *run*. Even though there is a one-to-one correspondence between the compositions of $n+1$ with odd summands and the binary strings of length n without odd runs of zeros, this one-to-one correspondence does not extend to quantities such as the number of levels, rises and drops.

We derive recurrence equations for several characteristics and express these quantities as functions of the Fibonacci and Lucas numbers, and also give their respective generating functions. In Section 2 we introduce our notation and state some basic facts about the Fibonacci and Lucas numbers that will be used in subsequent sections. Section 3 contains results on the total number of binary strings of length n without odd runs of zeros, the number of zeros and ones, the total number of runs, and the number of levels, rises and drops in all such strings. In addition, we show that the sum of the decimal values of all such binary strings of length n is congruent to 0 (mod 3) for even n, and F_{n+1} (mod 3) for odd n. Section 4 contains the corresponding results for palindromic binary strings of length n without odd runs of zeros. These are strings that read the same from left to right as from right to left. For the palindromic binary strings the results are similar as for the binary strings, but there are always separate formulas for odd and even n.

2 Notation and general observations

a_n	$=$	the total number of binary strings of length n without odd runs of zeros
$a_{n,0}, a_{n,1}$	$=$	the total number of binary strings of length n without odd runs of zeros ending in 0 and 1, respectively
z_n	$=$	the total number of zeros in all binary strings of length n without odd runs of zeros
w_n	$=$	the total number of ones in all binary strings of length n without odd runs of zeros
t_n	$=$	the total number of runs in all binary strings of length n without odd runs of zeros
v_n	$=$	the value of the sum of the a_n strings considered as the base 2 representation of decimal (base 10) integers

We use the same variable names with a $\tilde{}$ to denote the corresponding quantities for palindromic binary strings. The notation $G_{a_n}(x)$ is used for the generating function $\sum_{n=1}^{\infty} a_n x^n$ of the sequence $\{a_n\}_1^{\infty}$. F_n and L_n denote the n^{th} Fibonacci and Lucas number, respectively. Recall that explicit formulas for the n^{th} Fibonacci and Lucas numbers are given by the Binet forms

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2},$$

and that the generating functions are

$$G_{F_n}(x) = \frac{x}{1-x-x^2} \quad \text{and} \quad G_{L_n}(x) = \frac{x+2x^2}{1-x-x^2}.$$

Note that the sequence for the Lucas numbers starts with L_1, which is reflected in the generating function G_{L_n} - it does not contain the term for $L_0 = 2$. We will also need the generating functions of $\{nF_n\}_1^{\infty}$, $\{nL_n\}_1^{\infty}$, $\{2^nF_n\}_1^{\infty}$, and $\{2^nL_n\}_1^{\infty}$.

Lemma 1 *1.* $G_{nF_n}(x) = \frac{x+x^3}{(1-x-x^2)^2}$ *and* $G_{nL_n}(x) = \frac{x+4x^2-x^3}{(1-x-x^2)^2}$.
2. $G_{2^nF_n}(x) = G_{F_n}(2x) = \frac{2x}{(1-2x-4x^2)}$ *and* $G_{2^nL_n}(x) = G_{L_n}(2x) = \frac{2x+8x^2}{(1-2x-4x^2)}$.

Proof: 1. $G_{nF_n}(x) = x \cdot \frac{d}{dx}G_{F_n}(x)$ and $G_{nL_n}(x) = x \cdot \frac{d}{dx}G_{L_n}(x)$. (See for example [8], Eq. (2.2.2), p. 34.)
2. $G_{2^nF_n}(x) = \sum_{n=1}^{\infty} 2^n F_n x^n = \sum_{n=1}^{\infty} F_n (2x)^n = G_{F_n}(2x)$. Likewise for $G_{2^nL_n}(x)$. □

Both binary strings and palindromic binary strings without odd runs of zero can be created recursively from those of a shorter length. We will use these creation methods to derive recursions for the quantities of interest. For easier readability, we will leave out the specification "without odd runs of zeros" in the remainder of this article.

3 Results for binary strings

To create a binary string of length n, we can either append a 1 to a binary string of length $n-1$, or the string 00 to a binary string of length $n-2$. We will refer to this process as the *creation process*. First we look at the total number of such binary strings, and also count how many of these end in either 0 or 1.

Theorem 2 *1.* $a_n = F_{n+1}$ *for* $n \geq 1$.
2. $a_{n,0} = F_{n-1}$ *and* $a_{n,1} = F_n$.

Proof: From the creation process it is clear that $a_n = a_{n-1} + a_{n-2}$, the Fibonacci recurrence. Since $a_1 = 1$ and $a_2 = 2$, it follows that $a_n = F_{n+1}$. In addition, $a_{n,0} = a_{n-2} = F_{n-1}$ and $a_{n,1} = a_{n-1} = F_n$. □

Next we look at the total number of zeros that occur in the binary strings of length n. We will express z_n as a function of the n^{th} Fibonacci and Lucas numbers.

Theorem 3 *1.* $z_n = \frac{2}{5}nL_n - \frac{2}{5}F_n$ *for* $n \geq 1$ *and* $G_{z_n}(x) = \frac{2x^2}{(1-x-x^2)^2}$.
2. $w_n = \frac{2}{5}F_n + \frac{1}{10}nL_n + \frac{1}{2}nF_n$ *for* $n \geq 1$ *and* $G_{w_n}(x) = \frac{x}{(1-x-x^2)^2}$.

Proof: 1. We will show this proof in more detail, as many of the later proofs use the same argument and will only be sketched. From the creation process we get the following recurrence relation:

$$z_n = z_{n-1} + z_{n-2} + 2a_{n-2} = z_{n-1} + z_{n-2} + 2\frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}}. \tag{1}$$

When appending the 1, no new zeros are created, while two additional zeros are created for each of the a_{n-2} binary strings of length $n-2$. This difference equation has a solution of the form $z_n = z_n^{(h)} + z_n^{(p)}$. Since the associated homogeneous recurrence is the Fibonacci recurrence, it follows that $z_n^{(h)} = c_1\alpha^n + c_2\beta^n$ for some constants c_1 and c_2. The inhomogeneous part contains powers of α and β, hence $z_n^{(p)} = An\alpha^n + Bn\beta^n$ for some constants A and B (see for example [6]). Substituting $z_n^{(p)}$ into Eq. (1) and collecting only the terms that contain powers of α results in the following equation:

$$An\alpha^n = A(n-1)\alpha^{n-1} + A(n-2)\alpha^{n-2} + (2/\sqrt{5})\alpha^{n-1}. \tag{2}$$

Since α is a root of the Fibonacci recurrence, $An(\alpha^n - \alpha^{n-1} - \alpha^{n-2}) = 0$, and Eq. (2) simplifies to

$$0 = -A\alpha^{n-1} - 2A\alpha^{n-2} + (2/\sqrt{5})\alpha^{n-1}.$$

Dividing by α^{n-1}, substituting the value for α and solving for A gives $A = 2/5$. A similar computation for the terms that contain powers of β results in $B = 2/5$. Thus,

$$z_n = c_1\alpha^n + c_2\beta^n + \frac{2}{5}n(\alpha^n + \beta^n).$$

Using the initial conditions $z_1 = 0$ and $z_2 = 2$ results in $c_1 = -\frac{2\sqrt{5}}{25}$ and $c_2 = \frac{2\sqrt{5}}{25}$. Expressing sums and differences of powers of α and β as Lucas and Fibonacci numbers gives the desired result. From the expression for z_n it follows that $G_{z_n}(x) = (2/5)G_{nL_n}(x) - (2/5)G_{F_n}(x) = \frac{2x^2}{(1-x-x^2)^2}$ after simplification.
2. This can be proved similarly to part 1. Alternatively, $w_n + z_n = nF_{n+1}$. Substituting the solution for z_n and using $L_n - F_n = 2F_{n-1}$ (which can be easily shown by induction, or follows readily from the Binet forms for F_n and L_n) gives the result for w_n. The generating function G_{w_n} is computed in the same manner as G_{z_n}. □

Another quantity of interest is the number of runs in all binary strings of length n. Again, this can be expressed as a function of the Fibonacci and Lucas numbers.

Theorem 4 *$t_n = \frac{1}{10}(5L_n - 3F_n) + \frac{n}{5}(5F_n - L_n)$ for $n \geq 1$ and $G_{t_n}(x) = \frac{x - x^4}{(1-x-x^2)^2}$.*

Proof: Again we utilize the creation process. When creating the binary strings of length n, an additional run is created for every binary string of length $n-1$ that ends in 0, and also for every binary string of length $n-2$ that ends in 1, i.e. $a_{n-1,0} + a_{n-2,1} = F_{n-2} + F_{n-2}$ additional runs. Thus,

$$t_n = t_{n-1} + t_{n-2} + 2F_{n-2}, \text{ for } n \geq 3 \text{ with } t_1 = 1, t_2 = 2.$$

With the same method as in Theorem 3, we get $A = (-1 + \sqrt{5})/5$ and $B = (-1 - \sqrt{5})/5$. Using the initial conditions gives $c_1 = \frac{25-3\sqrt{5}}{50}$ and $c_2 = \frac{25+3\sqrt{5}}{50}$. Substituting the constants and expressing sums and differences of powers of α and β as Lucas and Fibonacci numbers gives the desired result, which also holds for $n = 1$ and $n = 2$. G_{t_n} is computed as the corresponding linear combination of G_{L_n}, G_{F_n}, G_{nL_n} and G_{nF_n}. □

In connection with runs, we can look at the total number of rises (switch from a 0 run to a 1 run), levels (within a run) and drops (switch from a 1 run to a 0 run). Since there are $n-1$ rises, levels or drops per binary string of length n, we get that

$$r_n + l_n + d_n = (n-1)F_{n+1}. \tag{3}$$

Furthermore, since the reverse of a binary string without odd runs of zeros is also a binary string without odd runs of zeros, we have that $r_n = d_n$. Thus we only need to derive the recurrence for the number of levels.

Theorem 5 *$l_n = \frac{1}{10}(3F_n - 5L_n) + \frac{n}{10}(7L_n - 5F_n)$ for $n \geq 1$ and $G_{l_n}(x) = \frac{2x^2 + x^4}{(1-x-x^2)^2}$.*

Proof: For $n \geq 3$, it follows from the creation process that

$$l_n = (l_{n-1} + a_{n-1,1}) + (l_{n-2} + 2a_{n-2,0} + a_{n-2,1})$$

because we get all the previous levels, plus one additional one whenever we append either a 1 to a sting of length $n-1$ that ends in 1, or a 00 to a string of length $n-2$ that ends in 1. Appending 00 to a string of length $n-2$ that ends in 0 gives rise to two additional levels. Using Theorem 2, we get

$$l_n = l_{n-1} + F_{n-1} + l_{n-2} + 2F_{n-3} + F_{n-2} = l_{n-1} + l_{n-2} + 3F_{n-3} + 2F_{n-2}.$$

This recurrence relation again has a solution of the form given in the proof of Theorem 3, and we get $A = (7-\sqrt{5})/10$ and $B = (7+\sqrt{5})/10$. Initial conditions $l_1 = 0$ and $l_2 = 2$ give $c_1 = -\frac{1}{2} + \frac{3\sqrt{5}}{50}$ and $c_2 = -\frac{1}{2} - \frac{3\sqrt{5}}{50}$. Collecting sums and differences of powers of α and β gives the desired result, which also holds for $n = 1$ and $n = 2$. The generating function is computed by simplifying the associated linear combination of the respective generating functions. □

Corollary 6 $r_n = d_n = \frac{1}{2}[(n-1)F_{n+1} - \frac{1}{10}(3F_n - 5L_n) - \frac{n}{10}(7L_n - 5F_n)]$ *for* $n \geq 1$ *and* $G_{r_n}(x) = \frac{x^3}{(1-x-x^2)^2}$.

Proof: The formula for r_n follows immediately from Eq. (3). Since $r_n = [(n-1)F_{n+1} - l_n]/2 = (n+1)F_{n+1}/2 - F_{n+1} - l_n/2$,we get

$$\begin{aligned}
G_{r_n}(x) &= \frac{1}{2}\sum_{n=1}^{\infty}(n+1)F_{n+1}x^n - \sum_{n=1}^{\infty}F_{n+1}x^n - \frac{1}{2}\sum_{n=1}^{\infty}l_n x^n \\
&= \frac{1}{2x}\sum_{k=2}^{\infty}kF_k x^k - \frac{1}{x}\sum_{k=2}^{\infty}F_k x^k - \frac{1}{2}G_{l_n}(x) \\
&= \frac{1}{2x}[G_{nF_n}(x) - x] - \frac{1}{x}[G_{F_n}(x) - x] - \frac{1}{2}G_{l_n}(x) \\
&= \frac{x^3}{(1-x-x^2)^2}.
\end{aligned}$$

□

Note that the generating function for r_n can be expressed as $G_{r_n}(x) = x(G_{F_n}(x))^2$, thus, the sequence for r_n is a shifted convolution of the Fibonacci sequence with itself.

Now we change focus a little and consider these stings as base 2 representations of decimal (base 10) integers. We will look at the sum of all the decimal values of the binary strings of length n, and look at their congruences mod 3. Instead of functions involving nL_n and nF_n we now get expressions that involve $2^n L_n$ and $2^n F_n$.

Theorem 7 $v_n = \frac{2^n}{11}(L_n + 7F_n) - \frac{1}{11}(L_n + 4F_n)$ *for* $n \geq 1$ *and* $G_{v_n}(x) = \frac{x}{(1-x-x^2)(1-2x-4x^2)}$.

Proof: Again we look at the creation process. We now have to determine what effect appending a 1 or a 00 has on the decimal value. Appending a 1 shifts the string to the left, hence results in a multiplication of the decimal value by 2, and then an addition of 1 from the appended 1. Appending 00 results in a shift to the left of two positions, hence results in multiplication of the decimal value by 4. As there are F_n binary strings of length $n-1$, we get the following recurrence:

$$v_n = 2v_{n-1} + F_n + 4v_{n-2}, \quad \text{with} \quad v_1 = 1 \text{ and } v_2 = 3.$$

In this case, the homogeneous recurrence relation has characteristic roots 2α and 2β. Thus, the general solution is of the form

$$v_n = c_1(2\alpha)^n + c_2(2\beta)^n + An\alpha^n + Bn\beta^n.$$

Now we proceed as in the proof of Theorem 3, which results in $A = -(5 + 4\sqrt{5})/55$ and $B = -(5 - 4\sqrt{5})/55$. Substituting the initial conditions and solving the resulting system of equations gives $c_1 = \frac{1}{11} + \frac{7\sqrt{5}}{55}$ and $c_2 = \frac{1}{11} - \frac{7\sqrt{5}}{55}$. Substituting these constants and grouping into sums and differences of powers of α and β gives the result for $n \geq 3$. However, this formula also holds for $n = 1$ and $n = 2$. The generating function is computed by taking the appropriate linear combination of the respective generating functions. □

Finally, we examine the following.

Theorem 8 *For even n, the decimal value of each individual binary string of length n is congruent to 0 (mod 3), and $v_n \equiv 0$ (mod 3) also. For odd n, the decimal value of each individual binary string of length n is congruent to 1 (mod 3), and $v_n \equiv F_{n+1}$ (mod 3).*

Proof: We show the congruence for the individual strings by induction. The result for v_n follows because there are F_{n+1} binary strings of length n. For $n = 1$, there is only one string, 1, whose value is congruent to 1 (mod 3). For $n = 2$, the only strings are 11 and 00 with decimal values of 3 and 0, respectively, and both of these are congruent to 0 (mod 3). We now assume the induction hypothesis and use the creation process. If we append a 1, then this corresponds to multiplication by 2 of the value of the string of length $n-1$ and addition of 1. Thus the string's value is (using the hypothesis) congruent to $2 \cdot 1$ (mod 3) + 1 (mod 3) $\equiv 0$ (mod 3) for even n, and congruent to $2 \cdot 0$ (mod 3) + 1 (mod 3) $\equiv 1$ (mod 3) if n is odd. If we append 00, then this corresponds to multiplication by $4 \equiv 1$ (mod 3) of the value for a string of length $n-2$ and the result follows. □

4 Results for palindromic binary strings without odd runs of zeros

We now derive the corresponding results for palindromic binary strings. Palindromic binary strings of length n can be created by either attaching a 1 to both ends of a palindromic binary string of length $n-2$ or 00 to both ends of a palindromic binary string of length $n-4$. We will refer to this way of creating palindromic binary strings as the *palindromic creation process.*

For odd n, we note that the middle digit must be a 1, as otherwise there would be an odd run of zeros in the center. Thus, a palindromic binary string of length $2k+1$ can also be thought of as a binary string of length k, concatenated with a 1, concatenated with the reverse of the binary string. This viewpoint will be referred to as the *explicit representation.*

Theorem 9 *1. $\tilde{a}_{2k} = F_{k+2}$ for $k \geq 1$ and $\tilde{a}_{2k+1} = F_{k+1}$ for $k \geq 0$ and $G_{\tilde{a}_n}(x) = \frac{x+2x^2+x^4}{(1-x^2-x^4)}$.*

2. $\tilde{a}_{n,0} = \tilde{a}_{n-4}$ and $\tilde{a}_{n,1} = \tilde{a}_{n-2}$.

Proof: 1. If $n = 2k$, then from the palindromic creation process we get

$$\tilde{a}_{2k} = \tilde{a}_{2k-2} + \tilde{a}_{2k-4} = \tilde{a}_{2(k-1)} + \tilde{a}_{2(k-2)},$$

which is once more the Fibonacci recurrence. Using the initial values, $\tilde{a}_2 = \tilde{a}_{2\cdot 1} = 2$ (for the strings 00 and 11), and $\tilde{a}_4 = \tilde{a}_{2\cdot 2} = 3$ (for the strings 0000, 1001, and 1111) gives that $\tilde{a}_{2k} = F_{k+2}$. If $n = 2k+1$, then the explicit representation gives $\tilde{a}_{2k+1} = a_k = F_{k+1}$, where the second equality follows from Theorem 2. For the generating function, we have to split up the series into odd and even terms, use the result of part 1, re-index, and simplify:

$$\begin{aligned} G_{\tilde{a}_n}(x) &= \sum_{k=0}^{\infty} \tilde{a}_{2k+1} x^{2k+1} + \sum_{k=1}^{\infty} \tilde{a}_{2k} x^{2k} = \sum_{k=0}^{\infty} F_{k+1} x^{2k+1} + \sum_{k=1}^{\infty} F_{k+2} x^{2k} \\ &= \frac{1}{x} \sum_{l=1}^{\infty} \tilde{F}_l (x^2)^l + \frac{1}{x^4} \sum_{l=3}^{\infty} \tilde{F}_l (x^2)^l \\ &= \frac{1}{x} G_{F_n}(x^2) + \frac{1}{x^4} [G_{F_n}(x^2) - (x^2) - (x^2)^2] = \frac{x+2x^2+x^4}{(1-x^2-x^4)}. \end{aligned}$$

2. This follows from the palindromic creation process. $\square$

Next we look at the number of zeros and ones in the palindromic binary strings of length n.

Theorem 10 *1.* $\tilde{z}_{2k} = -\frac{2}{5}F_k + 2kF_{k-1} + \frac{6}{5}kL_{k-1}$ *for* $k \geq 2$*;* $\tilde{z}_2 = 2$ *and* $\tilde{z}_{2k+1} = -\frac{4}{5}F_k + \frac{4}{5}kL_k$ *for* $k \geq 0$*.*
2. $\tilde{w}_{2k} = \frac{2}{5}F_k + 4kF_k - \frac{6}{5}kL_{k-1}$ *for* $k \geq 1$ *and* $\tilde{w}_{2k+1} = (2k+1)F_{k+1} + \frac{4}{5}F_k - \frac{4}{5}kL_k$ *for* $k \geq 0$*.*
3. $G_{\tilde{z}_n}(x) = \dfrac{2(x^2+x^4+2x^5+x^6)}{(1-x^2-x^4)^2}$ *and* $G_{\tilde{w}_n}(x) = \dfrac{x+2x^2+x^3+2x^4-x^5}{(1-x^2-x^4)^2}$.

Proof: 1. From the palindromic creation process, we get the following recursion:

$$\tilde{z}_{2k} = \tilde{z}_{2k-2} + \tilde{z}_{2k-4} + 4\tilde{a}_{2k-4} \text{ for } k \geq 3,$$

where the first two terms account for the "old" zeros, and the last term accounts for the four additional zeros for each palindromic binary string of length $n-4$. Defining $x_k = \tilde{z}_{2k}$ and using Theorem 9, part 1, we get

$$x_k = x_{k-1} + x_{k-2} + 4F_k.$$

Following the steps in the proof of Theorem 3 and using the initial conditions $x_1 = \tilde{z}_2 = 2$ and $x_2 = \tilde{z}_4 = 6$, we get the result for $x_k = \tilde{z}_{2k}$. Note that the formula also holds for $k = 2$.
If $n = 2k+1$, we get $\tilde{z}_{2k+1} = 2z_k$ from the explicit representation for $k \geq 1$. The result then follows from Theorem 3 and also holds for $k = 0$.
2. This follows from $\tilde{z}_n + \tilde{w}_n = n \cdot \tilde{a}_n$.
3. To compute $G_{\tilde{z}_n}(x)$, we split the series into odd and even terms, substitute the formulas from part 1 and adjust for the fact that the formula for even n only holds for $n \geq 4$:

$$\begin{aligned}
G_{\tilde{z}_n}(x) &= \sum_{l=0}^{\infty} \tilde{z}_{2l+1}x^{2l+1} + \sum_{l=1}^{\infty} \tilde{z}_{2l}x^{2l} = x\sum_{l=0}^{\infty} \tilde{z}_{2l+1}(x^2)^l + \sum_{l=1}^{\infty} \tilde{z}_{2l}(x^2)^l \\
&= x(-\frac{4}{5}G_{F_n}(x^2) + \frac{4}{5}G_{nL_n}(x^2)) \\
&\quad + \sum_{l=1}^{\infty}(-\frac{2}{5}F_l + 2lF_{l-1} + \frac{6}{5}lL_{l-1})(x^2)^l + \frac{12}{5}x^2 \\
&= x(-\frac{4}{5}G_{F_n}(x^2) + \frac{4}{5}G_{nL_n}(x^2)) - \frac{2}{5}G_{F_n}(x^2) \\
&\quad + 2x^2\sum_{l=1}^{\infty}((l-1)F_{l-1} + F_{l-1})(x^2)^{l-1} \\
&\quad + \frac{6}{5}x^2\sum_{l=1}^{\infty}((l-1)L_{l-1} + L_{l-1})(x^2)^{l-1} + \frac{12}{5}x^2.
\end{aligned}$$

Changing the summation index, replacing the series by the corresponding generating function, and then simplifying, gives the result.

Since $\tilde{w}_n = n \cdot \tilde{a}_n - \tilde{z}_n$, we get that $G_{\tilde{w}_n}(x) = x \cdot \frac{d}{dx} G_{\tilde{a}_n}(x) - G_{\tilde{z}_n}(x)$ (See for example [8], Eq. (2.2.2), p. 34.) $\square$

As with the binary strings, we can ask about the total number of runs of zeros and ones in the palindromic binary strings of length n.

Theorem 11 *For $k \geq 1$, $\tilde{t}_{2k} = \frac{4}{5} k L_k - \frac{1}{10}(17F_k + 5L_k)$ and $\tilde{t}_{2k+1} = \frac{1}{10}(15L_k - 21F_k) + \frac{k}{5}(10F_k - 2L_k)$, with $\tilde{t}_1 = 1$ and generating function $G_{\tilde{t}_n}(x) = \frac{1+x^3+2x^5+x^6+2x^8-3x^9}{(1-x^2-x^4)^2}$.*

Proof: With an argument similar to that in Theorem 4, and using Theorem 9, we get

$$\begin{aligned} \tilde{t}_n &= \tilde{t}_{n-2} + \tilde{t}_{n-4} + 2(\tilde{a}_{n-2,0} + \tilde{a}_{n-4,1}) \\ &= \tilde{t}_{n-2} + \tilde{t}_{n-4} + \begin{cases} 4F_{k-1} & \text{for } n = 2k \\ 4F_{k-2} & \text{for } n = 2k+1 \end{cases}, \end{aligned}$$

where the factor of two for the additional runs comes from the fact that we append on both sides. Making the substitution $x_k = \tilde{t}_{2k}$ or $x_k = \tilde{t}_{2k+1}$, we can now proceed as in the proof of Theorem 3. For $n = 2k$, we get $A = B = 4/5$, and using the initial conditions $\tilde{t}_2 = x_1 = 2$, $\tilde{t}_4 = x_2 = 5$ results in $c_1 = -\frac{1}{2} + \frac{17}{50}\sqrt{5}$ and $c_2 = -\frac{1}{2} - \frac{17}{50}\sqrt{5}$. For $n = 2k+1$, we get $A = \frac{2}{5}(\sqrt{5}-1)$ and $B = -\frac{2}{5}(\sqrt{5}+1)$, and using the initial conditions $\tilde{t}_3 = x_1 = 1$, $\tilde{t}_5 = x_2 = 4$ results in $c_1 = \frac{3}{50}(25-7\sqrt{5})$ and $c_2 = \frac{3}{50}(25+7\sqrt{5})$. The generating function is computed as in the proof of Theorem 10, with an adjustment for the value of $\tilde{t}_1$. $\square$

We now look at the total number of rises, levels and drops in all palindromic binary strings of length n. As before, we have $\tilde{r}_n = \tilde{d}_n$ and

$$\tilde{r}_n + \tilde{l}_n + \tilde{d}_n = (n-1)\tilde{a}_n = \begin{cases} (n-1)F_{k+2} & \text{for } n = 2k \\ (n-1)F_{k+1} & \text{for } n = 2k+1 \end{cases}. \tag{4}$$

Theorem 12 *For $k \geq 1$, $\tilde{l}_{2k} = \frac{1}{10}(5L_k - 17F_k) + \frac{k}{5}(L_k + 15F_k)$ and $\tilde{l}_{2k+1} = \frac{1}{5}(13F_k - 5L_k) + \frac{k}{5}(7L_k - 5F_k)$, with $\tilde{l}_1 = 0$ and generating function $G_{\tilde{l}_n}(x) = \frac{x^2(2+2x+3x^2+2x^3+3x^4-2x^5-x^6+2x^7)}{(1-x^2-x^4)^2}$.*

Proof: Similar to the proof of Theorem 5, and with the additional factors of 2 as in the proof of Theorem 11, we get for $n \geq 5$:

$$\tilde{l}_n = (\tilde{l}_{n-2} + 2\tilde{a}_{n-2,1}) + (\tilde{l}_{n-4} + 4\tilde{a}_{n-4,0} + 2\tilde{a}_{n-4,1}).$$

Using Theorem 9 and the Fibonacci recurrence, this reduces to

$$\tilde{l}_n = \tilde{l}_{n-2} + \tilde{l}_{n-4} + \begin{cases} 4F_k + 2F_{k-2} & \text{for } n = 2k \\ 4F_{k-1} + 2F_{k-3} & \text{for } n = 2k+1 \end{cases}.$$

Making the substitution $x_k = \tilde{l}_{2k}$ or $x_k = \tilde{l}_{2k+1}$, we can now proceed as in the proof of Theorem 3. For $n = 2k$, we get $A = \frac{4(4+\sqrt{5})}{5(1+\sqrt{5})}$ and $B = \frac{-4(4-\sqrt{5})}{5(1-\sqrt{5})}$. Using the initial conditions $\tilde{l}_2 = x_1 = 2$, $\tilde{l}_4 = x_2 = 7$ results in $c_1 = \frac{1}{2} - \frac{17}{50}\sqrt{5}$ and $c_2 = \frac{1}{2} + \frac{17}{50}\sqrt{5}$. For $n = 2k+1$, we get $A = \frac{4(4+\sqrt{5})}{5(3+\sqrt{5})}$ and $B = \frac{4(4-\sqrt{5})}{5(3-\sqrt{5})}$. Using the initial conditions $\tilde{l}_3 = x_1 = 2$, $\tilde{l}_5 = x_2 = 6$ results in $c_1 = -1 + \frac{13}{25}\sqrt{5}$ and $c_2 = -1 - \frac{13}{25}\sqrt{5}$, which gives the result for $\tilde{l}_n$ for $n \geq 5$. Note that the formula also holds for $2 \leq n \leq 4$. The generating function $G_{\tilde{l}_n}(x)$ is computed as

$$\begin{aligned} G_{\tilde{l}_n}(x) &= x\left(\frac{13}{5}G_{F_n}(x^2) - G_{L_n}(x^2) + \frac{7}{5}G_{nL_n}(x^2) - G_{nF_n}(x^2)\right) \\ &\quad + \frac{1}{2}G_{L_n}(x^2) - \frac{17}{10}G_{F_n}(x^2) + \frac{1}{5}G_{nL_n}(x^2) + 3G_{nF_n}(x^2) \end{aligned}$$

and simplification yields the result. □

Corollary 13 *For $k \geq 1$, $\tilde{r}_{2k} = \tilde{d}_{2k} = \frac{1}{2}(2k-1)F_{k+2} - \frac{1}{20}(5L_k - 17F_k) - \frac{k}{10}(L_k + 15F_k)$ and $\tilde{r}_{2k+1} = \tilde{d}_{2k+1} = kF_{k+1} - \frac{1}{10}(13F_k - 5L_k) - \frac{k}{10}(7L_k - 5F_k)$, with $\tilde{r}_1 = 0$ and generating function $G_{\tilde{r}_n}(x) = \dfrac{x^4(1+x+x^2+x^3+x^4-x^5)}{(1-x^2-x^4)^2}$.*

Proof: Follows immediately from Theorem 12 and Eq. (4). Since $\tilde{r}_n = \frac{1}{2}(n\tilde{a}_n - \tilde{a}_n - \tilde{l}_n)$, the generating function can be computed as $G_{\tilde{r}_n}(x) = \frac{1}{2}[x \cdot \frac{d}{dx}G_{\tilde{a}_n}(x) - G_{\tilde{a}_n}(x) - G_{\tilde{l}_n}(x)]$. □

Finally, we look at the palindromic binary strings as base 2 representations of decimal integers. First we give a formula for the sum of all the decimal values of the palindromic binary strings of length n.

Theorem 14 *For $k \geq 1$, $\tilde{v}_{2k} = \frac{3}{22}2^k(F_k - L_k) + \frac{2}{11}4^k(2L_k + 3F_k) - \frac{1}{22}(5L_k + 9F_k)$ and $\tilde{v}_{2k+1} = \frac{9}{22}2^k(F_k + L_k) + \frac{2}{11}4^k(L_k + 7F_k) - \frac{1}{11}(L_k + 4F_k)$ with*

$\tilde{v}_1 = 1$ *and generating function*
$G_{\tilde{v}_n}(x) = \frac{x(1+3x+3x^3-21x^4-30x^5-24x^7+16x^8)}{(1-x^2-x^4)(1-2x^2-4x^4)(1-4x^2-16x^4)}$.

Proof: We proceed as in the proof of Theorem 7, except that now we also have to take into account the changes on the left side. When appending a 1 to the right and left sides of a palindromic binary string of length $n-2$, we get an additional 2^{n-1} from the left side. When appending 00 on the right and left sides of a palindromic binary string of length $n-2$, the left side does not contribute anything to the decimal value. Thus, we get the following recursion for $n \geq 5$:

$$\tilde{v}_n = (2\tilde{v}_{n-2} + \tilde{a}_{n-2} + 2^{n-1} \cdot \tilde{a}_{n-2}) + 4 \cdot \tilde{v}_{n-4},$$

with initial conditions $\tilde{v}_1 = 1$, $\tilde{v}_2 = 3$, $\tilde{v}_3 = 7$, and $\tilde{v}_4 = 24$. Using Theorem 9, this reduces to

$$\tilde{v}_n = 2\tilde{v}_{n-2} + 4\tilde{v}_{n-4} + (2^{n-1}+1) \cdot \begin{cases} F_{k+1} & \text{for } n = 2k \\ F_k & \text{for } n = 2k+1 \end{cases}.$$

Making the usual substitution $x_k = \tilde{v}_{2k}$ and $x_k = \tilde{v}_{2k+1}$, respectively, we get a general solution of the form

$$x_k = c_1(2\alpha)^k + c_2(2\beta)^k + A\alpha^k + B\beta^k + C(4\alpha)^k + D(4\beta)^k$$

due to the factor of 2^{n-1} for the Fibonacci term. We proceed as in the proof of Theorem 3. For $n = 2k$, we get $A = -\frac{25+9\sqrt{5}}{110}$, $B = -\frac{25-9\sqrt{5}}{110}$, $C = \frac{40+12\sqrt{5}}{110}$, and $D = \frac{40-12\sqrt{5}}{110}$. The initial conditions give $c_1 = -\frac{3\sqrt{5}}{110}(1-\sqrt{5})$ and $c_2 = -\frac{3\sqrt{5}}{110}(1+\sqrt{5})$. For $n = 2k+1$, $A = -\frac{5+4\sqrt{5}}{55}$, $B = -\frac{5-4\sqrt{5}}{55}$, $C = \frac{10+14\sqrt{5}}{55}$, and $D = \frac{10-14\sqrt{5}}{55}$. Here, the initial conditions give $c_1 = \frac{9}{110}(5+\sqrt{5})$ and $c_2 = \frac{9}{110}(5-\sqrt{5})$. The generating function is computed as in the proof of Theorem 12. In particular,

$$\begin{aligned} G_{\tilde{v}_n}(x) &= x\left[\frac{9}{22}(G_{F_n}(2x^2) + G_{L_n}(2x^2)) + \frac{2}{11}(G_{L_n}(4x^2) + 7G_{F_n}(4x^2)) \right. \\ &\quad \left. -\frac{1}{11}(G_{L_n}(x^2) + 4G_{F_n}(x^2))\right] + \left[\frac{3}{22}(G_{F_n}(2x^2) - G_{L_n}(2x^2)) \right. \\ &\quad \left. +\frac{2}{11}(2G_{L_n}(4x^2) + 3G_{F_n}(4x^2)) - \frac{1}{22}(5G_{L_n}(x^2) + 9G_{F_n}(x^2))\right] \end{aligned}$$

which gives the result after substitution and simplification. □

Finally, we examine the following.

Theorem 15 *For $n = 2k$, the decimal value of each individual palindromic binary string of length n is congruent to 0 (mod 3), and $\tilde{v}_{2k} \equiv 0 \pmod 3$ also. For $n = 2k+1$, the decimal value of each individual palindromic binary string of length n is congruent to 1 (mod 3), and $\tilde{v}_{2k+1} \equiv F_{k+1} \pmod 3$.*

Proof: The proof follows along the lines of the proof of Theorem 8. We show the congruence for the individual terms by induction. The basic step for the induction follows from Theorem 8. We now assume the induction hypothesis and utilize the palindromic creation process. If we append 00 on both sides, then this corresponds to multiplication by $4 \equiv 1 \pmod 3$ of the value for a string of length $n-4$ and the result follows. If we append a 1 on each side of a palindromic binary string of length $n-2$, then this corresponds to multiplication by 2 of the value of the string of length $n-2$ and addition of $2^{n-1}+1$. Since $2^{2k} \pmod 3 \equiv 4^k \pmod 3 \equiv 1 \pmod 3$, we get for $n = 2k$, that the string's value is (using the hypothesis) congruent to $2\cdot 0 \pmod 3 + (2^{2k-1}+1) \pmod 3 \equiv (0+2\cdot 2^{2(k-1)}+1) \pmod 3 \equiv (0+2\cdot 1+1) \pmod 3 \equiv 0 \pmod 3$. For $n = 2k+1$, the string's value is (using the hypothesis) congruent to $2\cdot 1 \pmod 3 + (2^{2k}+1) \pmod 3 \equiv (2+1+1) \pmod 3 \equiv 1 \pmod 3$. The result for $\tilde{v}_{2k+1}$ follows because there are F_{k+1} palindromic binary strings of length $2k+1$. □

5 Connection to Compositions with Odd Summands

We now discuss the connection between binary strings of length $n-1$ and compositions of n with odd summands. We can visualize a composition of n as a board of size 1-by-n inches with potential cutting sites after each inch. At each potential cutting site, we either cut or do not cut, and the lengths of the resulting pieces will determine the summands in the composition. The cutting instruction for a composition of n can be given by a binary string of length $n-1$, where a 0 indicates "no cut", and a 1 indicates "cut", as shown in Figure 1.

Since an even number of "no cuts" results in a piece of odd length, there is a one-to-one correspondence between the compositions of n with only odd summands and the binary strings of length $n-1$ with no odd runs of zeros. Grimaldi [5] has investigated compositions with odd summands, and looked at the occurrences of individual summands, and the number of summands, "+"-signs, levels, rises and drops. However, the one-to-one correspondence between the total number of compositions of $n+1$

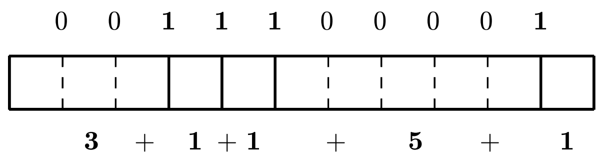

Figure 1: A composition and its binary string cutting instruction

with odd summands and the binary strings of n with no odd runs of zero does not extend automatically to these quantities. The only one-to-one correspondence is between s_{n+1}, the number of "+" signs in compositions of $n+1$ with odd summands and w_n, the number of 1's in binary strings of n. This can be easily seen since every 1 results in a cut which creates two pieces and therefore has to correspond to a "+" sign. The two formulas look somewhat different (Section 3 [5] and Theorem 3):

$$s_{n+1} = (-1/5)F_{n+1} + (1/5)(n+1)L_{n+1} \text{ for } n \geq 1$$

and

$$w_n = (2/5)F_n + (1/10)nL_n + (1/2)nF_n \text{ for } n \geq 1,$$

but can be shown to be equivalent by first using the fact that $L_{n+1} - F_{n+1} = 2F_n$ (see proof of Theorem 3, part 2) and then showing the remaining equality using the Binet forms for F_n and L_n.

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