Backbone Coloring for Graphs with Large Girths

Yuehua Bu* Daphne Der-Fen Liu † Xuding Zhu ‡

May 1, 2013

Abstract

For a graph G and a subgraph H (called *backbone* graph) of G, a *backbone k-coloring of G with respect to H* is a proper vertex coloring of G using colors from the set $\{1, 2, \ldots, k\}$, with an additional condition that colors for any two adjacent vertices in H must differ by at least two. The *backbone chromatic number of G over H*, denoted by $\mathrm{BBC}(G, H)$, is the smallest k of a backbone k-coloring admitted by G with respect to H. Broersma, Fomin, Golovach, and Woeginger [2] showed that $\mathrm{BBC}(G, H) \leq 2\chi(G) - 1$ holds for every G and H; moreover, for every n there exists a graph G with a spanning tree T such that $\chi(G) = n$ and the bound is sharp. To answer a question raised in [2], Miškuf, Škrekovski, and Tancer [17] proved that for any n there exists a triangle-free graph G with a spanning tree T such that $\chi(G) = n$ and $\mathrm{BBC}(G, T) = 2n - 1$. We extend this result by showing that for any positive integers n and l, there exists a graph G with a spanning tree T such that G has girth at least l, $\chi(G) = n$, and $\mathrm{BBC}(G, T) = 2n - 1$.

*Department of Mathematics, Zhejiang Normal University, China. Grant Number: NSFC No. 11271334 and ZJNSF No. Z6110786. Email: yhbu@zjnu.edu.cn

†Department of Mathematics, California State University, Los Angeles, Email: dliu@calstatela.edu

‡Department of Mathematics, Zhejiang Normal University, China. Grant Number: NSFC No. 11171310 and ZJNSF No. Z6110786. Email: xudingzhu@gmail.com

1 Introduction

Backbone coloring is a model for the channel assignment problem introduced by Hale [11]. The task in the channel assignment problem is to assign channels to a set of transmitters such that interference is avoided. Usually interference is divided into two types: strong interference and weak interference. The channels assigned to two transmitters with strong interference should be far apart, and channels assigned to two transmitters with weak interference should be distinct. A well studied graph theory model for the channel assignment problem is the distance-two labeling of graphs. We construct a graph where vertices represent the transmitters. If two vertices in the model graph are adjacent then stronger interference might occur between the two corresponding transmitters so the separation of these two channels needs to be at least two; and for two vertices that are distance two apart (that is, they are not adjacent but they share a common neighbor in G) then weak interference might occur between the two corresponding transmitters so they must receive different channels.

Backbone coloring of a graph is another model for the channel assignment problem, where edges in G are of two different types. Let H be a subgraph of G. An edge of G is either an edge of H which represents strong interference, or not an edge of H which represents weak interference. The subgraph H is called the *backbone* of G. In a backbone coloring of G with backbone H, colors assigned to a pair of vertices adjacent in H must be at least two apart, while vertices adjacent in G but not in H must get different colors. To be precise, a *backbone k-coloring of G with respect to H* is a function $f : V(G) \to \{1, 2, \dots, k\}$ such that the following are satisfied:

$$|f(u) - f(v)| \geq \begin{cases} 2 & \text{if } uv \in E(H); \\ 1 & \text{if } uv \in E(G) \setminus E(H). \end{cases}$$

The *backbone chromatic number of G over H*, denoted by $\mathrm{BBC}(G, H)$, is the minimum k for which there is a backbone k-coloring of G with respect to H.

For a graph G, the *square of G*, denoted by G^2, has $V(G)$ as the vertex set and $uv \in E(G^2)$ if $uv \in E(G)$ or there is a 2-path from u to v in

G. A *distance-two labeling* (also known as $L(2,1)$-*labeling*) is the same as a backbone coloring of G^2 with respect to the backbone G (however, a distance-two labeling allows 0 as a color while a backbone coloring uses only positive integers, hence a distance-two k-labeling of a graph G is a backbone $(k+1)$-coloring of G^2 with respect to G). Introduced by Griggs and Yeh [10], distance-two labeling has been studied extensively in the past three decades (cf. [5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 20, 21, 22]).

Backbone coloring was first introduced by Broersma et al. [1] and has been investigated widely by several authors in recent years. Broersma, Fomin, Golovach, and Woeginger [2] studied the BBC(G, H)-number when the backbone graph H is a spanning tree or a spanning path (if exists) of G. Miškuf, Škrekovski, and Tancer [18] proved that for a graph G with maximum degree Δ and backbone H being a d-degenerated subgraph of G, then $\mathrm{BBC}(G, H) \leq \Delta + d + 1$; moreover, if H is a matching then $\mathrm{BBC}(G, H) \leq \Delta + 1$.

Denote the chromatic number of a graph G by $\chi(G)$. By properly coloring the vertices of G from the set $\{1, 3, 5, \ldots, 2\chi(G)-1\}$, one obtains a backbone $(2\chi(G)-1)$-coloring of G with respect to any subgraph H. Therefore, for any graph G and any subgraph H of G, $\mathrm{BBC}(G, H) \leq 2\chi(G) - 1$. It was proved in [2] that for any positive integer n, there exists a graph G and a spanning tree T of G such that $\chi(G) = n$ and $\mathrm{BBC}(G, T) = 2\chi(G) - 1$. The graphs G used in the proof of this result are complete n-partite graphs, which contain many triangles.

An interesting question asked in [2] was whether there exits a constant c such that $\mathrm{BBC}(G, T) \leq \chi(G) + c$ holds for all triangle-free graphs G and spanning tree T of G. Miškuf, Škrekovski, and Tancer [17] answered this question in negative by showing that for any n there exists a triangle-free graph G with a spanning tree T such that $\chi(G) = n$ and $\mathrm{BBC}(G, T) = 2n-1$. The graphs constructed in [17], by a process similar to the construction of Mycielski graphs, are infinite and contain 4-cycles. Naturally, the authors raised the question regarding the existence of a graph G with large girth (i.e., the length of a shortest cycle in G) such that $\mathrm{BBC}(G, T) = 2\chi(G) - 1$ for some spanning tree T. We answer this question in positive.

Theorem 1. *For any positive integers n and l, there exists a graph G with girth greater than l and $\chi(G) = n$, and a spanning forest T of G such that* $\mathrm{BBC}(G, T) = 2n - 1$.

The proof of Theorem 1 is presented in the next two sections.

In Section 4 we discuss computational complexity for backbone colorings. Let T be a tree and G a spanning subgraph of T^2. We prove that if T has a bounded maximum degree, $\Delta(T) \leq c$ for some constant c, then there exists a polynomial-time algorithm to determine the exact value of $\mathrm{BBC}(G, T)$.

2 Construction of G and T

The construction of G and T in Theorem 1 will be based on the following result which seems to be a folklore. But we could not find a reference with the exact statement. For the completeness of the paper, we provide here an easy probabilistic proof.

Lemma 2. *For any positive integers n, l, m_0, for each $\delta > 0$, there is an n-partite graph G with partite sets $V_1, V_2, \ldots, V_n$ such that the following hold:*

- $|V_i| = m \geq m_0$.
- *G has $\chi(G) = n$ and girth greater than l.*
- *For any $1 \leq i \neq j \leq n$, for any $A \subset V_i, B \subset V_j$ with $|A|, |B| \geq \delta m$, there is an edge between A and B.*

Proof. Let $\epsilon = 1/(2l)$, m be a sufficiently large integer and $p = m^{-1+\epsilon}$. Let G be a random graph with vertex set $V = V_1 \cup V_2 \ldots \cup V_n$, where $|V_i| = 2m$, and uv is an edge of G with probability p for any $u \in V_i$, $v \in V_j$ $(1 \leq i < j \leq n)$. Let X be the random variable which is the number of cycles in G of length at most l. The expectation of X is

$$E(X) \leq \sum_{i=1}^{l} \frac{(2nm)_i}{2i} p^i \leq l(2nmp)^l = l(2n)^l m^{1/2},$$

where $(x)_i = x(x-1)\cdots(x-i+1)$. Hence

$$P(X > m) \leq \frac{E(X)}{m} \to 0, \text{ as } m \to \infty.$$

Let Y be the number of pairs of sets A, B such that $A \subset V_i, B \subset V_j$ with $i \neq j$, $|A|, |B| \geq \delta m$, and $E[A, B] = \emptyset$ (that is, there are no edges between A and B). The expectation of Y is

$$E(Y) \leq 2^{2nm}(1-p)^{(\delta m)^2} \leq e^{2nm - p\delta^2 m^2} = e^{2nm - \delta^2 m^{1+\epsilon}} \to 0, \text{ as } m \to \infty.$$

Hence

$$P(Y \geq 1) \leq E(Y) \to 0, \text{ as } m \to \infty.$$

So for m sufficiently large, $P(X > m) < 1/2$ and $P(Y \geq 1) < 1/2$, and hence $P((X \leq m) \wedge (Y = 0)) > 0$. This implies that there is a graph G in which the number of short cycles is less than m and for any pair $A \subset V_i, B \subset V_j$ with $i \neq j$, $|A|, |B| \geq \delta m$, we have $E[A, B] \neq \emptyset$. Delete m vertices from each V_i (for $i = 1, 2, \ldots, n$) so that each short cycle intersects the deleted vertices. The resulting graph G' has girth at least $l + 1$.

Without loss of generality, we may assume that $\delta \leq 1/n$. Then $\chi(G') = n$. For otherwise, let c be an $(n-1)$-coloring of G'. For each $1 \leq i \leq n$, let c_i be a color used by at least δm vertices of V_i. Then $c_i = c_j$ for some $i \neq j$. However, there is an edge connecting a vertex of color c_i in V_i and a vertex in V_j of color $c_j = c_i$, a contradiction. This completes the proof of Lemma 2. $\square$

Let $0 < \delta < \frac{1}{2n^2 n! 2^n}$ be a fixed real number, and assume $\kappa = \delta m$ is an integer. Assume G is a graph satisfying the condition of Lemma 2. We shall construct a spanning forest T of G so that $\mathrm{BBC}(G, T) = 2n - 1$. Let $b = 2n^2\kappa$.

Lemma 3. *Assume G is a graph satisfying the condition of Lemma 2. If $1 \leq i < j \leq n$, $A \subseteq V_i$, $B \subseteq V_j$ and $|A| = |B| \geq \kappa$, then there is a matching M from A to B of size $|M| \geq |A| - \kappa$.*

Proof. We prove the lemma by induction on $|A| - \kappa$. If $|A| - \kappa = 0$, then there is nothing to prove. Assume $|A| - \kappa \geq 1$. By our assumption, there is an edge $e = xy$ with $x \in A$ and $y \in B$. By induction hypothesis, there is a matching M' from $A - \{x\}$ to $B - \{y\}$ of size $|M'| \geq |A| - \kappa - 1$. Then $M = M' \cup \{xy\}$ is a matching from A to B of size $|M| \geq |A| - \kappa$. $\square$

Let π be a permutation of $\{1, 2, \ldots, n\}$. We construct a forest F_π recursively. Note that in the following although the notations might seem complicated (as we need to, in every iteration, keep track of all vertices and edges included in the forest), the idea itself is quite simple. We shall adopt several figures to facilitate visuality of the notations.

Let $V(F_\pi)$ be the vertex set of F_π. We shall denote by $V_{\pi,i}$ the set $V(F_\pi) \cap V_{\pi(i)}$.

At the beginning we choose two sets $V^0_{\pi,n} \subseteq V_{\pi(n)}$ and $V^0_{\pi,n-1} \subseteq V_{\pi(n-1)}$ so that $|V^0_{\pi,n}| = |V^0_{\pi,n-1}| = b$. By Lemma 3, there is a matching $M_{\pi,n-1}$ of size $b - \kappa$ from $V^0_{\pi,n-1}$ to $V^0_{\pi,n}$. Let $F_{\pi,n-1}$ denote the subgraph induced by the edges in $M_{\pi,n-1}$. Let $V^1_{\pi,j} = V(F_{\pi,n-1}) \cap V_{\pi(j)}$ for $j = n-1, n$. See Figure 1 for a drawing of $F_{\pi,n-1}$.

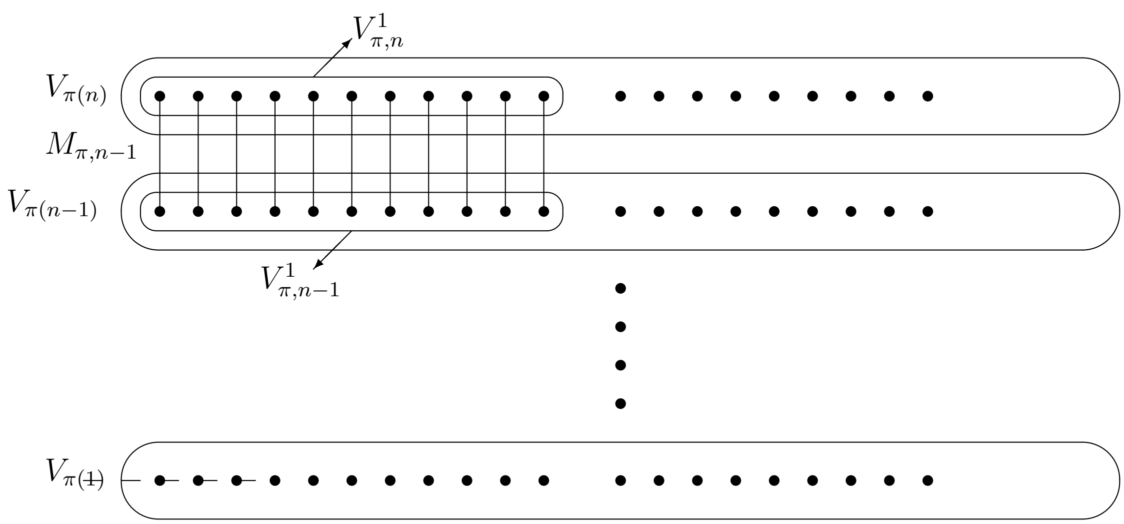

Figure 1: $F_{\pi,n-1}$ has $b - \kappa$ edges (components).

Next, choose a set $V^1_{\pi,n-2} \subseteq V_{\pi(n-2)}$ so that $|V^1_{\pi,n-2}| = \sum_{j=n-1}^{n} |V^1_{\pi,j}|$. Let $M_{\pi,n-2}$ be a matching from $V^1_{\pi,n-2}$ to $V^1_{\pi,n-1} \cup V^1_{\pi,n}$ so that for $j = n-1, n$, at least $|V^1_{\pi,j}| - \kappa$ vertices of $V^1_{\pi,j}$ are incident to edges of $M_{\pi,n-2}$. Such a matching exists by Lemma 3.

Let $F'_{\pi,n-2}$ be the forest induced by $M_{\pi,n-2} \cup F_{\pi,n-1}$. Each component of $F'_{\pi,n-2}$ is viewed as a rooted tree with a root in $V^1_{\pi,n}$. Delete from $F'_{\pi,n-2}$ those components which contain a vertex not incident to an edge in $M_{\pi,n-2}$. The resulting forest is denoted by $F_{\pi,n-2}$. Let $V^2_{\pi,j} = V(F_{\pi,n-2}) \cap V_{\pi(j)}$ for $j = n-2, n-1, n$. See Figure 2 for a drawing of $F_{\pi,n-2}$.

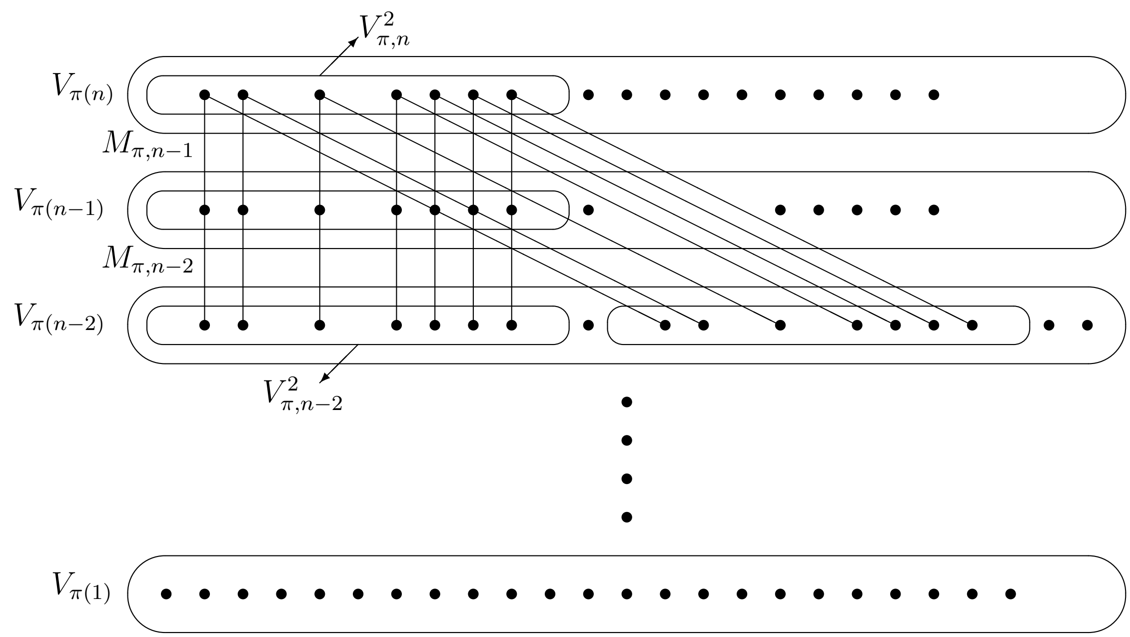

Figure 2: $F_{\pi,n-2}$ has at most $b - 3\kappa$ components. Note some vertices or edges were deleted from $F_{\pi,n-1}$.

Note that there are at most 2κ vertices of $F_{\pi,n-1}$ not incident to an edge of $M_{\pi,n-2}$. Hence at most 2κ components of $F'_{\pi,n-2}$ are deleted. In particular, we have

$$|V^2_{\pi,n-1}| = |V^2_{\pi,n}| \geq b - 3\kappa,$$

and

$$|V^2_{\pi,n-2}| = |V^2_{\pi,n-1}| + |V^2_{\pi,n}|.$$

Continue this process. In general, assuming $i \le n-2$, the forest $F_{\pi,n-i}$ is constructed with the following properties:

(1) $F_{\pi,n-i}$ has at least $b - \frac{i(i+1)}{2}\kappa$ components.

(2) Let $V^i_{\pi,j} = V(F_{\pi,n-i}) \cap V_{\pi(j)}$ for $j = n-i, n-i+1, \ldots, n$. Then for $n-i \le j < j' \le n$, any vertex in $V^i_{\pi,j'}$ has a unique neighbour (in the forest $F_{\pi,n-i}$) in $V^i_{\pi,j}$.

(3) $|V^i_{\pi,n-1}| = |V^i_{\pi,n}| \ge b - \frac{i(i+1)}{2}\kappa$; and for $j = n-i, n-i+1, \ldots, n-2$,

$$|V^i_{\pi,j}| = \sum_{j'=j+1}^{n} |V^i_{\pi,j'}|.$$

We claim that the above properties are kept in the next step. Choose a set $V^i_{\pi,n-i-1} \subseteq V_{\pi(n-i-1)}$ so that $|V^i_{\pi,n-i-1}| = \sum_{j=n-i}^{n} |V^i_{\pi,j}|$. Let $M_{\pi,n-i-1}$ be a matching from $V^i_{\pi,n-i-1}$ to $\cup_{j=n-i}^{n} V^i_{\pi,j}$ so that for $j = n-i-1, n-i, \ldots, n$, at least $|V^i_{\pi,j}| - \kappa$ vertices of $V^i_{\pi,j}$ are incident to edges of $M_{\pi,n-i-1}$. Such a matching exists by Lemma 3.

Let $F'_{\pi,n-i-1}$ be the forest induced by $M_{\pi,n-i-1} \cup F_{\pi,n-i}$. Delete from $F'_{\pi,n-i-1}$ those components which contains a vertex not incident to an edge in $M_{\pi,n-i-1}$. The resulting forest is denoted by $F_{\pi,n-i-1}$. Let $V^{i+1}_{\pi,j} = V(F_{\pi,n-i-1}) \cap V_{\pi(j)}$ for $j = n-i-1, n-i, \ldots, n$.

Since there are at most $(i+1)\kappa$ vertices of $F_{\pi,n-i}$ not incident to an edge of $M_{\pi,n-i-1}$, at most $(i+1)\kappa$ components of $F'_{\pi,n-i-1}$ are deleted. Therefore (1) and (3) are satisfied for $F_{\pi,n-i-1}$. From the construction of $F_{\pi,n-i}$, we know that if $n-i-1 \le j < j' \le n$, then any vertex in $V^{i+1}_{\pi,j'}$ has a unique neighbour in $V^{i+1}_{\pi,j}$ (in $F_{\pi,n-i-1}$). So, (2) is satisfied.

At the end of the process, let $F_\pi = F_{\pi,1}$ and $V_{\pi,1} = V^{n-1}_{\pi,1}$. The forest F_π has the following properties:

- The number of components of F_π is at most b and at least $b - \frac{n(n-1)}{2}\kappa$.

- If $1 \leq j < j' \leq n$, then each vertex in $V_{\pi,j'}$ has a unique neighbour (in the forest F_π) in $V_{\pi,j}$.

Observe that each component of F_π contains one vertex in $V_{\pi,n}$ and 2^{n-j-1} vertices of $V_{\pi,j}$ for $1 \leq j \leq n-1$. So each component of F_π has 2^{n-1} vertices, and hence F_π has at most $2^{n-1}b$ vertices.

Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$. We choose the forests F_π so that for $\pi, \pi' \in S_n, \pi \neq \pi'$, F_π and $F_{\pi'}$ are vertex disjoint. Since $m \geq 2^n n! b$, by Lemma 3, it can be verified easily that the family of vertex disjoint forests $\{F_\pi : \pi \in S_n\}$ can be constructed. Let $T = \cup_{\pi \in S_n} F_\pi$.

In the next section, we prove $\mathrm{BBC}(G, T) = 2n - 1$.

3 Proof of Theorem 1

Since $\chi(G) = n$, we know that $\mathrm{BBC}(G, T) \leq 2n - 1$. Assume to the contrary that $\mathrm{BBC}(G, T) \leq 2n - 2$. Let c be a backbone $(2n-2)$-coloring for G with respect to T.

For $1 \leq i \leq n$, let $C_{i,1}$ be the set of colors that have been used by at least k vertices in V_i. That is,

$$C_{i,1} = \{a : |c^{-1}(a) \cap V_i| \geq \kappa\}.$$

Because there are $2n - 2$ colors and $|V_i| = m > (2n-2)\kappa$, so $|C_{i,1}| \geq 1$ for all i.

By Lemma 3, $C_{i,1} \cap C_{j,1} = \emptyset$ for $i \neq j$. As $| \cup_{i=1}^n C_{i,1}| \leq 2n - 2$, there exists some $1 \leq s_1 \leq n$ such that $|C_{s_1,1}| = 1$. Assume $C_{s_1,1} = \{a_1\}$.

Now we consider the restriction of c to the subgraph of G induced by

$$U_1 = \cup_{\pi \in S_n, \pi(1) = s_1} V(F_\pi).$$

Let

$$W_1 = \{v \in U_1 \cap V_{s_1} : c(v) = a_1\}.$$

For each π with $\pi(1) = s_1$, delete from F_π those components which contain a vertex in $V_{s_1} \setminus W_1$. Denote the resulting forest by $F'_{\pi,1}$. Since $|c^{-1}(i) \cap V_{s_1}| <$

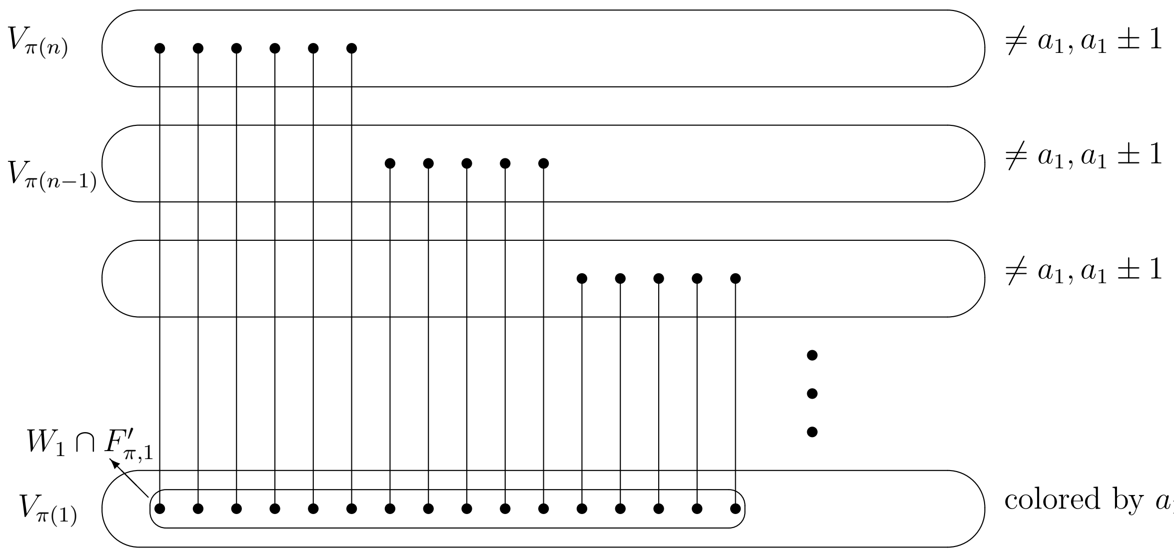

Figure 3: A partial drawing of $F'_{\pi,1}$ with edges connecting vertices in W_1 and other partite sets.

κ for $i \neq a_1$, we know that for each π with $\pi(1) = s_1$,

$$|V_{\pi,1} \setminus W_1| \leq (2n-3)\kappa.$$

Hence at most $(2n-3)\kappa$ components of F_π are deleted, so $F'_{\pi,1}$ has at least $b - (\frac{n(n-1)}{2} + (2n-3))\kappa$ components.

By our construction of F_π, for $i \neq s_1$, every vertex

$$v \in V_i \cap (\cup_{\pi \in S_n, \pi(1)=s_1} V(F'_{\pi,1}))$$

is adjacent to a vertex $u \in W_1$ by an edge in $F'_{\pi,1}$. Hence $c(v) \notin \{a_1 - 1, a_1, a_1 + 1\}$. See Figure 3 for in illustration of $F'_{\pi,1}$.

Similarly, for $1 \leq i \leq n$, $i \neq s_1$, let $C_{i,2}$ be the set of colors that have been used by at least κ vertices in $\cup_{\pi \in S_n, \pi(1)=s_1} V(F'_{\pi,1}) \cap V_i$. That is,

$$C_{i,2} = \{a : |c^{-1}(a) \cap V_i \cap (\cup_{\pi \in S_n, \pi(1)=s_1} V(F'_{\pi,1}))| \geq \kappa\}.$$

Since $|V_i \cap (\cup_{\pi \in S_n, \pi(1)=s_1} V(F'_{\pi,1}))| \geq b - n^2\kappa \geq n^2\kappa$, each $C_{i,2}$ is not empty. By the observation above, $C_{i,2} \cap \{a_1 - 1, a_1, a_1 + 1\} = \emptyset$. Moreover, by Lemma

3, $C_{i,2} \cap C_{j,2} = \emptyset$ for $i \neq j$. As

$$|\cup_{i \neq s_1, 1 \leq i \leq n} C_{i,2}| \leq |\{1, 2, \ldots, 2n-2\} - \{a_1 - 1, a_1, a_1 + 1\}| \leq 2n - 4,$$

there exists an index $s_2 \neq s_1$ such that $|C_{s_2,2}| = 1$. Assume $C_{s_2,2} = \{a_2\}$. Note, $|a_2 - a_1| \geq 2$.

Now we consider the restriction of c to the subgraph of G induced by

$$U_2 = \cup_{\pi \in S_n, \pi(1)=s_1, \pi(2)=s_2} V(F'_{\pi,1}).$$

Let

$$W_2 = \{v \in U_2 \cap V_{s_2} : c(v) = a_2\}.$$

For each π with $\pi(1) = s_1$ and $\pi(2) = s_2$, delete from $F'_{\pi,1}$ those components which contain a vertex in $V_{\pi,2} \setminus W_2$. Denote the resulting forest by $F'_{\pi,2}$. Similarly, $|c^{-1}(i) \cap V_{s_2}| < \kappa$ for $i \neq a_2$, and $|c^{-1}(i) \cap V_{s_2} \cap V(F'_{\pi,1})| = 0$ for $i \in \{a_1 - 1, a_1, a_1 + 1\}$. Hence

$$|(V_{s_2} \cap V(F'_{\pi,1})) \setminus W_2| \leq (2n-5)\kappa,$$

and at most $(2n-5)\kappa$ components of $F'_{\pi,1}$ are deleted. So $F'_{\pi,2}$ has at least $b - (\frac{n(n-1)}{2} + (2n-3) + (2n-5))\kappa$ components.

In general, assume $1 \leq q \leq n-1$ and we have chosen $s_1, s_2, \ldots, s_q$. Let $S_{n;s_1,s_2,\ldots,s_q}$ be the set of permutations $\pi \in S_n$ with $\pi(j) = s_j$ for $j = 1, 2, \ldots, q$. Assume for each $\pi \in S_{n;s_1,s_2,\ldots,s_q}$, we have constructed a forest $F'_{\pi,q}$ such that the following are true:

(1) For $j = 1, 2, \ldots, q$, vertices in $V(F'_{\pi,q}) \cap V_{\pi(j)}$ are colored by a_j.

(2) $|a_j - a_{j'}| \geq 2$ if $j \neq j'$.

(3) $F'_{\pi,q}$ has at least $b - (\frac{n(n-1)}{2} + (2n-3) + (2n-5) + \cdots + (2n-(2q+1)))\kappa$ components.

(4) If $1 \leq j < j' \leq n$, then each vertex in $V_{\pi,j'} \cap V(F'_{\pi,q})$ has a unique neighbour in $V_{\pi,j} \cap V(F'_{\pi,q})$ (in $F'_{\pi,q}$).

Now we show that the above properties also hold in the next step. By (1) above, for $i \neq s_1, s_2, \ldots, s_q$, every vertex

$$v \in V_i \cap \left(\cup_{\pi \in S_{n;s_1,s_2,\ldots,s_q}} V(F'_{\pi,q})\right)$$

is adjacent to a vertex of color $a_1, a_2, \ldots, a_q$ by an edge in $F'_{\pi,q}$. Hence $c(v) \notin \cup_{j=1}^{q}\{a_j - 1, a_j, a_j + 1\}$.

For $1 \leq i \leq n$, $i \neq s_1, s_2, \ldots, s_q$, let

$$C_{i,q+1} = \{a : |c^{-1}(a) \cap V_i \cap (\cup_{\pi \in S_{n;s_1,s_2,\ldots,s_q}} V(F'_{\pi,q})| \geq \kappa\}.$$

Similarly, it is easy to verify that $C_{i,q+1} \neq \emptyset$, and as observed above,

$$C_{i,q+1} \cap \left(\cup_{j=1}^{q}\{a_j - 1, a_j, a_j + 1\}\right) = \emptyset.$$

Moreover, by Lemma 3, $C_{i,q+1} \cap C_{j,q+1} = \emptyset$ for $i \neq j$. As

$$|\cup_{i \notin \{s_1,s_2,\ldots,s_q\}, 1 \leq i \leq 2n-2} C_{i,2}| \leq |\{1, 2, \ldots, 2n-2\} - \cup_{j=1}^{q}\{a_j-1, a_j, a_j+1\}| \leq 2n-2q-2,$$

there exists an index s_{q+1} such that $|C_{s_{q+1},q+1}| = 1$. Assume $C_{s_{q+1}} = \{a_{q+1}\}$. Note, $|a_{q+1} - a_j| \geq 2$ for all $j = 1, 2, \ldots, q$. So, (2) holds.

We consider the restriction of c to the subgraph of G induced by

$$U_{q+1} = \cup_{\pi \in S_{n;s_1,s_2,\ldots,s_q,s_{q+1}}} V(F'_{\pi,q}).$$

Let

$$W_{q+1} = \{v \in U_{q+1} \cap V_{s_{q+1}} : c(v) = a_{q+1}\}.$$

For each $\pi \in S_{n;s-1,s_2,\ldots,s_{q+1}}$, delete from $F'_{\pi,q}$ those components which contain a vertex in $V_{\pi,q+1} \setminus W_{q+1}$. Denote the resulting forest by $F'_{\pi,q+1}$. Similarly, we have

$$|V_{\pi,q+1} \setminus W_{q+1}| \leq (2n - (2q + 3))\kappa.$$

Hence at most $(2n - (2q + 3))\kappa$ components of $F'_{\pi,q}$ are deleted. So, (3) is satisfied for $F'_{\pi,q+1}$. In addition, it can be easily seen that (1) and (4) also hold for $F'_{\pi,q+1}$.

Assume we have chosen $s_1, s_2, \ldots, s_{n-1}$ and colors $a_1, a_2, \ldots, a_{n-1}$, and for the permutation π with $\pi(i) = s_i$ for $i = 1, 2, \ldots, n-1$, we have constructed the forest $F'_{\pi,n-1}$. By the discussion above, $F'_{\pi,n-1}$ has at least $b - (\frac{n(n-1)}{2} + (n-1)^2)\kappa > 1$ components. Hence, $F'_{\pi,n-1}$ is not empty. However, vertices in $F'_{\pi,n-1} \cap V_{\pi(n)}$ cannot be colored by any color in $\cup_{j=1}^{n-1}\{a_j - 1, a_j, a_j + 1\} = \{1, 2, \ldots, 2n-2\}$. This is an obvious contradiction, which completes the proof of Theorem 1. $\square$

References

[1] H.J. Broersma, F.V. Fomin, P.A. Golovach, G.J. Woeginger, Backbone colorings for networks, Preceedings of the 29th Internatiolal Workshop on Graph-Theoretic Concepts in Computer Science, LNCS 2880 (2003) 131 – 142.

[2] H.J. Broersma, F.V. Fomin, P.A. Golovach, G.J. Woeginger, Backbone colorings for graphs: Trees and path backbones, J. Graph Theory 55 (2007) 137 – 152.

[3] H.J. Broersma, J. Fujisawa, I. Marchal, D. Paulusma, A.N.M. Salman, K. Yoshimoto, λ-Backbone colorings along pairwise disjoint stars and matchings, Discrete Math. 309 (2009) 5596 – 5609.

[4] H.J. Broersma, L. Marchal, D. Paulusma, Backbone colorings along stars and matchings in split graphs: Their span is close to the chromatic number, Discuss. Math. Graph Theory 29 (2009) 143 – 162.

[5] G. Chang, C. Ke, D. Kuo, D. Liu, R. Yeh, A generalized distance two labeling of graphs, Disc. Math. 220 (2000) 57 – 66.

[6] G. Chang, D. Kuo, The $L(2,1)$-labelling problem on graphs, SIAM J. Disc. Math. 9 (1996) 309 – 316.

[7] J. Georges and D. Mauro, On sizes of graphs labeled with a condition at distance two, J. Graph Theory 22 (1996) 47 – 57.

[8] J. Georges, D. Mauro, and M. Stein, Labeling products of complete graphs with a condition at distance two, SIAM J. Disc. Math. 14 (2000) 28 – 35.

[9] J. Georges, D. Mauro, and M. Whittlesey, Relating path covering to vertex labellings with a condition at distance two, Disc. Math. 135 (1994) 103 – 111.

[10] J. Griggs and R. Yeh, Labeling graphs with a condition at distance 2, SIAM J. Disc. Math. (1992) 586 – 595.

[11] W. Hale, Frequency assignment: theory and applications, Proc. IEEE, 68 (1980) 1497 – 1514.

[12] J. van den Heuvel, R. Leese and M. Shepherd, Graph labelling and radio channel assignment, J. Graph Theory 29 (1998) 263 – 283.

[13] S. Klavžar and S. Špacapan, The Δ^2-conjecture for $L(2,1)$-labelings is true for directed and strong products of graphs, IEEE Trans. Circuits and Systems II 53 (2006) 274 -V 277.

[14] D. Král' and R. Škrekovski, A Theorem about the Channel Assignment, SIAM Journal on Discrete Mathematics 16 (2003) 426 – 437.

[15] D. Kuo and J.-H. Yan, On L(2, 1)-labelings of Cartesian products of paths and cycles, Discrete Math. 283 (2004) 137 V- 144.

[16] D. Liu and X. Zhu, Circular distance two labeling and the λ number for outer planar graphs, SIAM J. on Discrete Mathematics, 19 (2004) 281 - 293.

[17] J. Miškuf, R. Škrekovski, M. Tancer, Backbone colorings and generalized Mycielski's graphs, SIAM J. Disc. Math. 23 (2009) 1063 – 1070.

[18] J. Miškuf, R. Škrekovski, M. Tancer, Backbone colorings of graphs with bounded degree, Discrete Applied Math. 158 (2010) 534 – 542.

[19] J. Nešetřil and X. Zhu, On sparse graphs with given colorings and homomorphisms, J. Combin. Theory, Series B 90 (2004) 161 – 172.

[20] D. Sakai, *Labeling chordal graphs with a condition at distance two*, SIAM J. Disc. Math., 7 (1994), 133–140.

[21] C. Schwarz and D. Sakai Troxell, $L(2,1)$-Labelings of products of two cycles, Disc. Applied Math. 154 (2006) 1522 – 1540.

[22] M. Whittlesey, J. Georges, and D. auro, On the λ-number of Q_n and related graphs, SIAM J. Disc. Math. 8 (1995) 499 – 506.