Tiling an m-by-n Area with Squares of Size up to k-by-k (m$\leq$ 5)

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Abstract Formulas for the number of tilings $T_{m,n}$ of an m-by-n rectangle with square tiles of size up to k-by-k, for $m \leq 5$ are derived. Two cases are considered: 1) tilings that use only squares of size 1-by-1 and 2-by-2, and 2) tilings that use squares of size up to k-by-k, where k = min{m, n}. Using the idea of basic blocks (tilings that cannot be vertically split into smaller rectangles), a general recursive formula for the number of tilings is obtained. Explicit formulas are proved for the number of basic blocks of size m-by-k for m = 3 and m = 4 (both cases) and for $T_{3,n}$ for case 1. For m = 5, the number of basic blocks of size 5-by-k is determined recursively.

Keywords: Tiling of Rectangles, Square Tiles, Fibonacci Sequence, Jacobsthal Sequence

1. Introduction

The question to be discussed in this paper is a generalization of the problem of tiling a 1-by-n or 2-by-n rectangle with Cuisinaire rods ("c-rods"), color-coded rods of lengths 1 cm to 10 cm (1 cm = white, 2 cm = red). C-rods are used to help students with an intuitive understanding of concepts related to whole numbers, as well as geometry. In addition to their usefulness in K-8, the number of tilings of a rectangle of size 1-by-n with white and red c-rods is connected to the Fibonacci numbers. This connection can be used to give geometrical proofs of various relationships for this well-known sequence [1].

Extensions to the tiling question of a 1-by-n area with white and red c-rods have been investigated by a number of authors. Using the same types of c-rods, but covering a 2-by-n rectangle has been explored by Brigham et. al [2]. A generalization, allowing c-rods of length less than or equal to k for tiling 2-by-n and 3-by-n rectangles has been investigated by Hare [4,5] and by Hare and Chinn [6].

In this paper, the tiles used are squares, rather than c-rods. We will discuss two cases for tiling m-by-n rectangles, namely 1) to allow 1-by-1 and 2-by-2 tiles only; and 2) to allow tiles of size up to k-by-k, where k = min{m,n}. The approach taken is based on basic blocks, comparable to the indecomposable blocks utilized in [2]. A recursive formula for the number of tilings based on basic blocks is derived (Lemma 1). In the case m = 3 with tiles of size up to 2-by-2, an explicit formula for the number of tilings results (Theorem 1). In all other cases, the recursive structure remains, but explicit formulas for the number of basic blocks are derived. The sole exception is the case m = 5, with tiles of size up to 5-by-5, where the number of basic blocks is given by a recursive formula (Theorem 6).

2. The Trivial Cases: m = 1 and m = 2

In the case of tiling a 1-by-n rectangle, there is only one possible tiling, namely the one where all the tiles are of size 1-by-1. When m = 2, 1-by-1 and 2-by-2 squares can be used for tiling. However, they cannot be mixed and matched in every possible fashion. If a 1-by-1 tile is used somewhere in the rectangle, then it has to be paired vertically with another 1-by-1 tile. Thus we get tilings that are sequences consisting of either two vertically stacked 1-by-1 tiles or the 2-by-2 tile. Due to the symmetry induced by the pairing of the two 1-by-1 tiles, tilings of size 2-by-n using squares have a one-to-one correspondence to tilings of a 1-by-n rectangle with white and red Cuisinaire rods (see Figure 1).

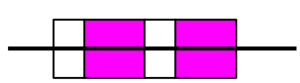

Figure 1

Formulas for the number of such tilings are well-known and can be found for example in [1]. A recursive relation is the basic idea for finding the number of these tilings. Rectangles of size 1-by-(n+1) are formed by either attaching a 1-by-1 rod to the left of a 1-by-n tiling, or by attaching a 1-by-2 rod to the left of a tiling of size 1-by-(n-1). In this manner, a recursion very much like the one for the Fibonacci numbers results, except that the initial conditions are "shifted". The generalization of this approach will be illustrated in the next section.

3. The Notion of Basic Blocks

When tiling rectangles with squares, the approach is similar. The basic idea is to form larger tilings by putting together tilings of smaller sizes in a specific way. The role of the 1-by-1 and 1-by-2 Cuisinare rods in the above example is now played by *basic* (or *indecomposable) blocks*. A basic block is a tiling that cannot be split (vertically) into two or more smaller rectangular pieces without cutting some of the squares. Figure 2 shows an example for tilings of size 4-by-3. The tiling on the left represents a basic block, whereas the second one does not qualify as a basic block, as it can be split into a 4-by-2 and a 4-by-1 rectangle.

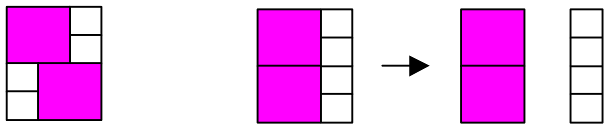

Figure 2

4. Notation

We now introduce some notation and conventions used throughout the paper. When talking about an m-by-n rectangle, m refers to the height and n refers to the width of the rectangle. Figure 3 shows the shadings which will be used for the different sizes of squares:

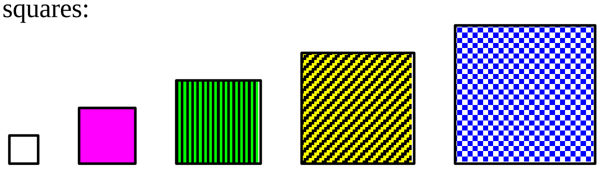

Figure 3

Furthermore, we let

$T_{m,n}$ = the number of tilings of an m-by-n rectangle with squares of size 1-by-1 and 2-by-2

$\tilde{T}_{m,n}$ = the number of tilings of an m-by-n rectangle with squares of size up to k-by-k, where $k = \min\{m, n\}$

$B_{m,n}$ = the number of basic blocks of size m-by-n using squares of size 1-by-1 and 2-by-2

$\tilde{B}_{m,n}$ = the number of basic blocks of size m-by-n using squares of size up to k-by-k, where $k = \min\{m, n\}$

F_m = the m^{th} Fibonacci number, with $F_1 = F_2 = 1$, and $F_m = F_{m-1} + F_{m-2}$

Note that $\tilde{T}_{m,n} = T_{m,n}$ and $\tilde{B}_{m,n} = B_{m,n}$ for $n = 1$ and $n = 2$.

5. The General Recursive Formula

The main idea is to combine a basic block of size m-by-k with a tiling of size m-by-(n-k) where $k \le n$. (We will always assume that the basic block is added from the left.) As all the tilings of size m-by-(n-k) are different, the newly created tilings are also different. On the other hand, each tiling starts with a basic block on the left (namely the first section that cannot be vertically separated). Thus, creating larger tilings by adding a basic block to the left of appropriate tilings will create all possible tilings. Double counting as shown in Figure 4 cannot happen, as the second way to create the given tiling does not follow the algorithm. (The tiling added from the left is not a basic block.)

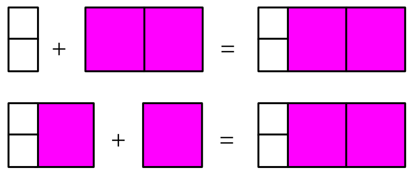

Figure 4

We will illustrate the general recursive formula for the case m = 2. The only basic blocks are of size 2-by-1 (two vertically stacked 1-by-1 tiles) and size 2-by-2 (the 2-by-2 tile), hence $B_{2,1} = B_{2,2} = 1$. We can create tilings of size 2-by-n in one of two ways: either by adding the basic block of size 2-by-1 to a tiling of size 2-by-(n-1) or by adding the basic block of size 2-by-2 to a tiling of size 2-by-(n-2). This leads to the recursive formula

$$T_{2,n} = B_{2,1}\, T_{2,\,n-1} + B_{2,2}\, T_{2,\,n-2} = T_{2,\,n-1} + T_{2,\,n-2}$$

The initial conditions are $T_{2,1} = 1$ and $T_{2,2} = 2$ (either a 2-by-2 tile or the tiling consisting of all 1-by-1 tiles). Therefore, the number of tilings of a 2-by-n rectangle is given by the shifted Fibonacci numbers, i. e. ,

$$T_{2,n} = F_{n+1}. \qquad (1)$$

In general, the number of tilings of size m-by-n can be created by combining a basic block of size m-by-k with a tiling of size m-by-(n-k), or it can consist of just a basic block of size m-by-n (if there are any). The same argument applies for tilings that allow squares of size up to k-by-k. If we define $T_{m,0} = \tilde{T}_{m,0} = 1$, then we get the following lemma:

Lemma 1: *The number of tilings of an m-by-n rectangle is given by*

$$T_{m,n} = \sum_{k=1}^{n} B_{m,k} T_{m,n-k} \quad \text{and} \quad \tilde{T}_{m,n} = \sum_{k=1}^{n} \tilde{B}_{m,k} \tilde{T}_{m,n-k} .$$

To make this formula useful, we need to determine the number of basic blocks of size m-by-k. We will first work on tilings that only use 1-by-1 and 2-by-2 squares, and then look at the general case.

6. Tilings with Squares of Size 1-by-1 and Size 2-by-2

6.1 Basic Blocks of Size m-by-1 and m-by-2

Before looking at specific values for m, we will state a fact that holds true independent of the value for m:

$$T_{m,1} = B_{m,1} = 1 \qquad \text{for all values of } m. \qquad (2)$$

(Each tiling consists of vertically stacked tiles of size 1-by-1.) Furthermore, we can easily derive a general formula for $B_{m,2}$.

The number of basic blocks of size m-by-2 is closely related to the number of tilings of size 2-by-m. The two types of tilings both cover the same size rectangle, viewed either vertically or horizontally. However, any basic block of width 2 must have at least one tile of size 2-by-2; therefore, the one tiling consisting of all 1-by-1 tiles has to be excluded. Together with (1), this leads to

$$B_{m,2} = T_{2,m} - 1 = F_{m+1} - 1. \quad (3)$$

On the other hand, using a combinatorial argument (see [1]) for the number of ways to place either a 2-by-2 tile or two adjacent 1-by-1 tiles, one can also derive that

$$B_{m,2} = \sum_{r=1}^{\lfloor m/2 \rfloor} \binom{m-r}{r}. \quad (4)$$

Combining (3) and (4) leads to a well-known formula for the Fibonacci numbers (see for example [3]).

6.2 The Case m = 3

The first step is to determine the number of basic blocks and the initial conditions for the recursive formula. As we only look at tiles of size 1-by-1 and 2-by-2, there are no basic blocks of size 3-by-n for $n > 2$. (To make basic blocks of larger size, there needs to be an interlocking mechanism, for which two 2-by-2 tiles are needed.) Thus, the only basic blocks for $m = 3$ are given in Figure 5, resulting in $B_{3,1} = 1$ and $B_{3,2} = 2$.

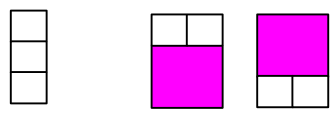

Figure 5

The general recursive formula (Lemma 1) reduces to

$$T_{3,n} = B_{3,1}\, T_{3,n-1} + B_{3,2}\, T_{3,n-2} = T_{3,n-1} + 2\, T_{3,n-2} \quad \text{for } n > 2, \quad (5)$$

with initial conditions $T_{3,1} = 1$ and $T_{3,2} = 3$ (the two basic blocks and the tiling consisting of only 1-by-1 tiles). However, this recursive formula has an explicit solution.

Theorem 1*: The number of tilings of a 3-by-n rectangle with squares of size 1-by-1 and 2-by-2 is given by*

$$T_{3,n} = \left(2^{n+1} - (-1)^{n+1}\right)/3$$

<u>Proof</u>:

The formula holds for the initial conditions:

$$n = 1: \; T_{3,1} = \left(2^2 - 1\right)/3 = 1$$
$$n = 2: \; T_{3,2} = \left(2^3 + 1\right)/3 = 3.$$

Furthermore, the recurrence relation (5) is identical to the recurrence for the Jacobsthal sequence (A001045 [7] or M2482 [8]), with $T_{3,1} = a(2)$ and $T_{3,2} = a(3)$. Thus,

$$T_{3,n} = a(n+1) = \left(2^{n+1} - (-1)^{n+1}\right)/3.$$

$\boxtimes$

Table 1 lists the first ten values of $T_{3,n}$.

n	1	2	3	4	5	6	7	8	9	10
$T_{3,n}$	1	3	5	11	21	43	85	171	341	683

Table 1

6.3 The Case m = 4

Again we start by determining the number of basic blocks. Now the height of the blocks is large enough to allow for an interlocking effect (for $n \geq 3$) that creates basic blocks of any length. From equations (2) and (3) in Section 6.1 we know that $B_{4,1} = 1$ and $B_{4,2} = 4$. Figure 6 shows the four possible basic blocks of size 4-by-2.

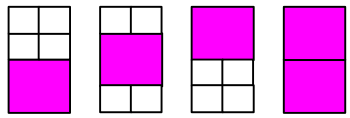

Figure 6

For $n > 2$, $B_{4,n} = 2$, as can be seen easily from the tilings shown in Figure 7.

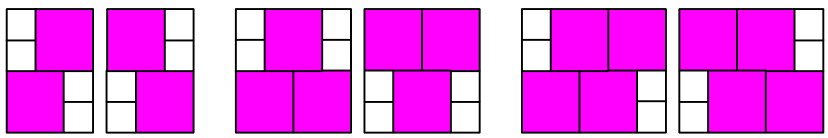

Figure 7

Since the 2-by-2 tiles have to be placed in an interlocking pattern to ensure the tiling forms a basic block, there are only two possibilities, depending on whether the leftmost 2-by-2 tile is located at the bottom or at the top. (A placement of the 2-by-2 tile in the middle position will only result in a basic block of width 2.) After the initial placement, there is only one way to extend the basic block for each of the two cases, giving exactly two basic blocks of size 4-by-*n*, $n > 2$.

Thus, we get the following result:

Theorem 2: *The number of tilings of a 4-by-n rectangle with squares of size 1-by-1 and 2-by-2 is given by the recursive formula*

$$T_{4,n} = T_{4,n-1} + 4T_{4,n-2} + 2\sum_{k=0}^{n-3} T_{4,k} \ ,$$

with $T_{4,0} = T_{4,1} = 1$, and $T_{4,2} = 5$.

<u>Proof:</u>

The recursive formula and re-indexing of the sum leads to the formula for $T_{4,n}$. The number of tilings of size 4-by-2 consist of the 4 basic blocks of that size and the tiling of all 1-by-1 squares.

$\boxtimes$

Table 2 lists the first ten values of $T_{4,n}$.

n	1	2	3	4	5	6	7	8	9	10
$T_{4,n}$	1	5	11	35	93	269	747	2,115	5,933	16,717

Table 2

6. 4 The Case m = 5

This case differs from the cases for $m = 3$ and $m = 4$ in that the number of basic blocks is not as easily determined as before. Here we will develop a recursive formula for the number of basic blocks of size 5-by-n, for $n > 3$. From (3) we know that $B_{5,2} = 7$; the respective tilings are given in Figure 8.

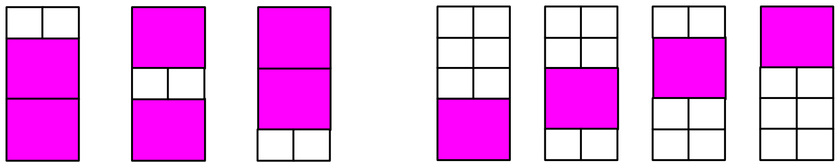

Figure 8

We now create basic blocks of size 5-by-(n+1) from those of size 5-by-n. We do this by placing a 2-by-2 tile over two vertically stacked tiles of size 1-by-1 in the rightmost column, and then filling up the remaining empty spaces in the new column with 1-by-1 tiles. Note the following: In order for a tiling to be a basic block of height 5, each vertical column must contain at least one 2-by-2 tile. In addition, any column but the leftmost and rightmost must contain two 2-by-2 tiles (to produce the interlocking effect).

Looking at the first three basic blocks in Figure 8, it is clear that they cannot be extended to basic blocks of size 5-by-3. For the other four basic blocks, there are two different possibilities, shown in Figure 9, depending on the structure of the rightmost column. If the 1-by-1 tiles are split into a single tile and a group of two tiles, then there is only one way to extend the basic block to the next larger size. If, on the other hand, there are three (vertically) stacked 1-by-1 tiles, then the 2-by-2 tile can be placed in two different ways. Thus, two of the basic blocks of size 5-by-2 produce one larger basic block of size 5-by-3, and the other two produce two basic blocks each.

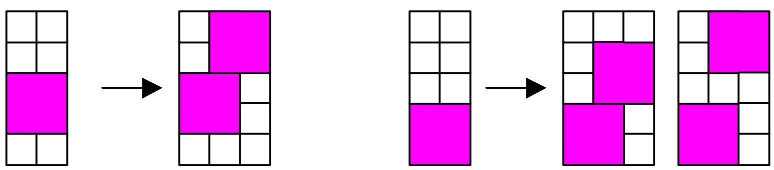

Figure 9

Altogether, we get a total of 6 basic blocks of size 5-by-3, grouped in Figure 10 according to their "ancestor" in Figure 8.

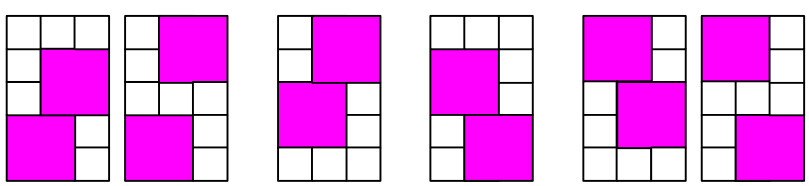

Figure 10

These are the only possible basic blocks of size 5-by-3, as exactly two 2-by-2 tiles in interlocking positions are needed. Let's look at this extension process in a little bit more detail. The difference between the two extensions of basic blocks shown in Figure 9 is the rightmost column. If we just look at this column, we can distinguish between two types:

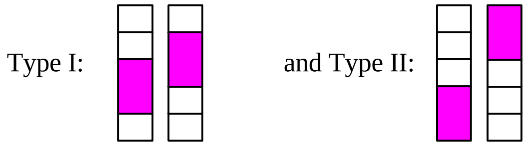

Figure 11

Each basic block of type I creates one new basic block of the next size, and those of type II produce two new basic blocks. But we also have to keep track of the type of the newly created blocks. Figure 12 shows what happens:

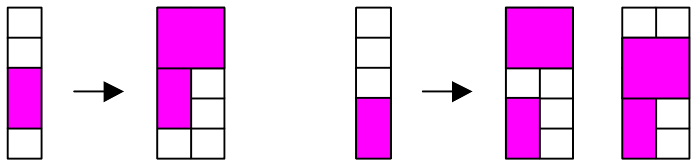

Figure 12

A type I block creates a type II block (as the single square and the 2-by-2 tile both get extended with 1-by-1 tiles to form a group of three consecutive 1-by-1 tiles). The same reasoning explains why one of the basic blocks created from a type II block also has to be of type II. (This is the one where the extending 2-by-2 tile is placed not adjacent to the existing 2-by-2 tile.) However, the second basic block created from a type II basic block is of type I. Altogether, we have the following lemma:

Lemma 2: *For n > 1, a type I basic block of size 5-by-n creates one type II basic block of size 5-by-(n+1); a type II basic block of size 5-by-n creates one type I and one type II basic block of size 5-by-(n+1).*

If we denote the number of basic blocks of type I and type II of size 5-by-*n* by $B^{I}_{5,n}$ and $B^{II}_{5,n}$, respectively, then we get the following formula for the number of basic blocks for $n > 2$ (as we have only type I and type II basic blocks):

$$B_{5,n} = B^{I}_{5,n} + B^{II}_{5,n} \qquad (6)$$

Now we can use Lemma 2 to determine formulas for the number of basic blocks of each type. Basic blocks of type I are created from basic blocks of type II only, hence

$$B^{I}_{5,n+1} = B^{II}_{5,n} \qquad (7)$$

On the other hand, basic blocks of type II are created from basic blocks of type I and type II. Each such block creates exactly one new block of type II. These blocks are all different, because their first n-1 columns were different. Therefore,

$$B^{II}_{5,n+1} = B^{I}_{5,n} + B^{II}_{5,n} = B_{5,n}. \qquad (8)$$

Combining (6), (7), and (8) results in the following formula for the number of basic blocks.

Lemma 3*: The number of basic blocks of size 5-by-m is given by*

$$B_{5,n} = 2 \cdot F_{n+1} \quad \text{for } n > 2,$$

with $B_{5,2} = 7$, and $B_{5,1} = 1$.

Proof:
For $n = 1$ and $n = 2$, the result is true by (2) and (3). Figure 10 shows that $B_{5,3} = 6 = 2 \cdot F_4$.

For $n = 4$, we recall that $B^I_{5,2} = B^{II}_{5,2} = 2$ (Figure 8). By Lemma 2, each of the two basic blocks of type I produces one type II basic block of size 5-by-3, and each of the two basic blocks of type II produces one basic block of type I and type II. Thus ,

$$B^I_{5,3} = 2 \quad \text{and} \quad B^{II}_{5,3} = 2 + 2 = 4 \ .$$

We now apply Lemma 2 once more to determine the number of basic blocks of type I and type II of size 5-by-4. Thus,

$$B^I_{5,4} = 4 \quad \text{and} \quad B^{II}_{5,4} = 4 + 2 = 6 \ .$$

This implies that $B_{5,4} = 4 + 6 = 10 = 2 \cdot F_5$.

For $n > 4$,

$$\begin{aligned} B_{5,n} &= B^I_{5,n} + B^{II}_{5,n} \\ &= B^{II}_{5,n-1} + B_{5,n-1} && \text{by (7) and (8)} \\ &= B_{5,n-2} + B_{5,n-1} && \text{by (8)} . \end{aligned}$$

As this is the recursion for the Fibonacci numbers, and the factor 2 carries through, the result follows.

$\boxed{\wedge}$

This leads to the following recursive formula for the number of tilings of size 5-by-*n*:

***Theorem 3**: The number of tilings of a 5-by-n rectangle with squares of size 1-by-1 and 2-by-2 is given by the recursive formula*

$$T_{5,n} = \sum_{k=1}^{n} B_{5,k} \, T_{5,n-k}$$

with

$$B_{5,n} = 2 \cdot F_{n+1} \quad \text{for} \quad n > 2,$$

where $B_{5,2} = 7$, $B_{5,1} = 1$, and $T_{5,1} = T_{5,0} = 1$.

Table 3 contains the first ten values for both $B_{5,n}$ and $T_{5,n}$.

n	1	2	3	4	5	6	7	8	9	10
$B_{5,n}$	1	7	6	10	16	26	42	68	110	178
$T_{5,n}$	1	8	21	93	314	1,213	4,375	16,334	59,925	221,799

Table 3

7. Tilings with Squares up to Size k-by-k, with k = min{n, m}

7.1 The Case m = 3 (Squares up to Size 3-by-3)

If we now allow tiles of size 3-by-3, only basic blocks of width larger than 2 can change. Obviously, there is one new basic block of size 3-by-3. However, as the 3-by-3 tile cannot be combined with any other tile to form an interlocking pattern due to height constraints, this is the only additional basic block. Thus, we have:

$$\tilde{B}_{3,1} = 1,\ \tilde{B}_{3,2} = 2, \text{ and } \tilde{B}_{3,3} = 1.$$

Using the general recursive formula, we obtain the following result:

***Theorem 4**: The number of tilings of a 3-by-n rectangle with squares of up to size 3-by-3 is given by*

$$\tilde{T}_{3,n} = \tilde{T}_{3,n-1} + 2\,\tilde{T}_{3,n-2} + \tilde{T}_{3,n-3}$$

with initial conditions $\tilde{T}_{3,0} = \tilde{T}_{3,1} = 1$ and $\tilde{T}_{3,2} = 3$ *as before.*

Even though there is only one additional basic block, the dynamics for the number of tilings changes quite dramatically. Table 4 gives the first ten values of $\tilde{T}_{3,n}$:

n	1	2	3	4	5	6	7	8	9	10
$\tilde{T}_{3,n}$	1	3	6	13	28	60	129	277	595	1,278

Table 4

7. 2 The Case m = 4 (Squares up to Size 4-by-4)

As in the case m = 3, the only changes in the number of basic blocks can occur for n > 2. Figure 13 shows the two additional blocks of width 3 and the one additional block of width 4.

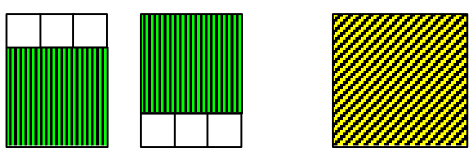

Figure 13

Neither the 3-by-3 nor the 4-by-4 tile can be used to make larger basic blocks through interlocking. Thus, these three are the only additional basic blocks, leading to:

$$\begin{aligned}&\tilde{B}_{4,1} = 1,\ \tilde{B}_{4,2} = 4,\ \tilde{B}_{4,3} = 2+2 = 4,\ \tilde{B}_{4,4} = 2+1 = 3,\\ &\text{and } \tilde{B}_{4,n} = 2 \quad \text{for } n > 4.\end{aligned} \tag{9}$$

We have the following result:

Theorem 5*: The number of tilings of a 4-by-n rectangle with squares of size up to 4-by-4 is given by*

$$\tilde{T}_{4,n} = \tilde{T}_{4,n-1} + 4\,\tilde{T}_{4,n-2} + 4\,\tilde{T}_{4,n-3} + 3\,\tilde{T}_{4,n-4} + 2\sum_{k=0}^{n-5} \tilde{T}_{4,k}\ ,$$

$$\textit{with}\ \tilde{T}_{4,0} = \tilde{T}_{4,1} = 1,\ \tilde{T}_{4,2} = 5,\ \textit{and}\ \tilde{T}_{4,3} = 13.$$

Proof:
The result for $n > 3$ follows from the general recursive formula and a re-indexing of the sum. Furthermore, $\tilde{T}_{4,m} = T_{4,m}$ for m = 0, 1, and 2, since the same tiles are being used. Using the general recursive formula (Lemma 1) and (9) leads to

$$\tilde{T}_{4,3} = \sum_{k=1}^{3} \tilde{B}_{4,k}\ \tilde{T}_{4,n\text{-}k} = 1\cdot 5 + 4\cdot 1 + 4 = 13\,.$$

$\square$

Table 5 displays the first ten values of $\tilde{T}_{4,n}$:

n	1	2	3	4	5	6	7	8	9	10
$\tilde{T}_{4,n}$	1	5	13	40	117	348	1,029	3,049	9,028	26,738

Table 5

7.3 The Case m = 5 (Squares up to Size 5-by-5)

In addition to the basic blocks made up from just 1-by-1 and 2-by-2 tiles, we now also allow 3-by-3, 4-by-4, and 5-by-5 tiles. Like in the case m = 3, the 4-by-4 and 5-by-5 tiles will only lead to basic blocks of their respective widths, as they cannot be extended in an interlocking fashion in combination with 2-by-2 tiles. Thus, any regular pattern for basic blocks will start for $n > 5$. We will first derive the number of basic blocks for $n \le 5$.

Lemma 4: *The number of basic blocks of size 5-by-n, for n ≤ 5, using tiles of size up to 5-by-5, are as follows:*

$$\tilde{B}_{5,1} = 1,\ \ \tilde{B}_{5,2} = 7,\ \ \tilde{B}_{5,3} = 13,\ \ \tilde{B}_{5,4} = 20,\ \text{and}\ \ \tilde{B}_{5,5} = 35.$$

Proof:
As the first additional basic blocks show up for n = 3, $\tilde{B}_{5,1} = B_{5,1} = 1$, and $\tilde{B}_{5,2} = B_{5,2} = 7$. For n = 3, we can now utilize 3-by-3 tiles, which can be placed in one of three positions, as shown in Figure 14.

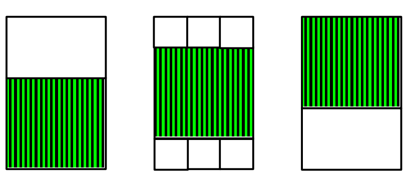

Figure 14

The first and third constellations are symmetrical, and the (white) 2-by-3 rectangle can be tiled in any way, i.e., in $T_{2,3}$ = 3 ways. The second tiling in Figure 14 shows the only possibility if the 3-by-3 tile is placed in the middle position. Therefore, there are $2 \cdot 3 + 1 = 7$ basic blocks of size 5-by-3 using a 3-by-3 tile. Using Lemma 3, we have that $\tilde{B}_{5,3} = B_{5,3} + 7 = 13$.

If n = 4, then we get additional basic blocks formed by either using a 3-by-3 tile together with an interlocking 2-by-2 tile, or by using the 4-by-4 tile. If we use a combination of 2-by-2 and 3-by-3 basic blocks, then there are only four possible positions for the interlocking 2-by-2 and 3-by-3 tiles, as shown in Figure 15.

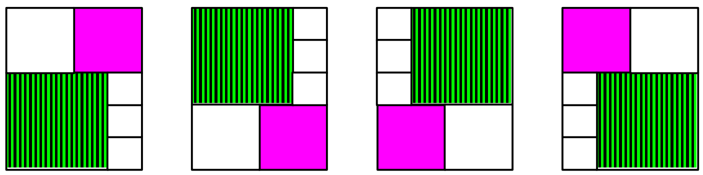

Figure 15

The (white) 2-by-2 square next to the interlocking 2-by-2 tile can be tiled in 2 ways (all 1-by-1 tiles or one 2-by-2 tile). Thus, there are altogether 8 basic blocks made up from 2-by-2 and 3-by-3 tiles. In addition, we can use a 4-by-4 tile, which will lead to 2 basic blocks of size 5-by-4. Altogether, $\tilde{B}_{5,4} = B_{5,4} + 8 + 2 = 10 + 10 = 20$.

Finally, for n = 5, the additional basic blocks are either made from a combination of 2-by-2 and 3-by-3 tiles or consist of the 5-by-5 tile. In the former case, we need one 3-by-3 tile and two 2-by-2 tiles to create an interlocking structure. There are six possibilities to place the 3-by-3 tile: two of these allow for exactly one placement of the 2-by-2 tiles, whereas the other four allow for two different placements each. Figure 16 shows the different possible positions of these tiles.

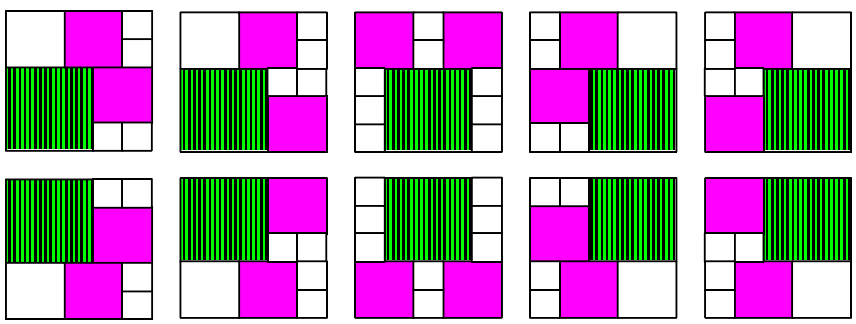

Figure 16

All but the two middle basic blocks in Figure 16 still contain a (white) 2-by-2 square that can be tiled in two ways. Thus, there are $8 \cdot 2 + 2 = 18$ basic blocks containing both 2-by-2 and 3-by-3 tiles. Altogether, using Lemma 3, $\tilde{B}_{5,5} = B_{5,5} + 18 + 1 = 16 + 19 = 35$.

Now that we have established the number of basic blocks for $n \leq 5$, we will look at a mechanism to create basic blocks of size 5-by-(n+1) from basic blocks of size 5-by-n. We will use an approach similar to the one used in Section 6.4. However, since tiles of size 3-by-3 can be used in the extension, we need to look at the **two**, instead of just one, rightmost columns of a basic block. Lemma 5 establishes the possible configurations for the last two columns of a basic block.

Lemma 5: *For n > 5, the last two columns of basic blocks of size* 5-*by-n must be of one of the five types shown in Figure* 17.

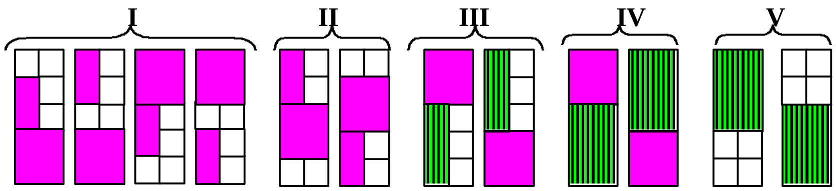

Figure 17

<u>Proof:</u>
The last two columns must contain a 2-by-2 or a 3-by-3 tile in order to ensure the tiling is a basic block. If the last two columns contain only 1-by-1 and 2-by-2 tiles, then the second to last column must contain two 2-by-2 tiles, and the last one can have only one 2-by-2 tile (due to the interlocking nature). The rest of the column is filled by three 1-by-1 tiles, which can be either grouped together (type I) or split into a pair of 2 and a single one (type II).

If both of the last columns are covered with a 3-by-3 tile (which necessarily is either on the top or bottom to allow for an interlocking extension), then the remaining 2-by-2 rectangle can be tiled with either a 2-by-2 tile (type IV) or four 1-by-1 tiles (type V). If the 3-by-3 tile only covers the next to last column, then it must be paired with a 2-by-2 tile in an interlocking fashion. The only such possibility is given by type III.

We can now establish how basic blocks of size 5-by-(n+1) can be created from basic blocks of size 5-by-n. We will denote the number of basic blocks of size 5-by-n which are of type I by $\tilde{B}^{I}_{5,n}$ (and likewise for the other four types).

Lemma 6: *The number of basic blocks of size 5-by-(n+1) for n $\geq$ 5 is given by*

$$\tilde{B}_{5,n+1} = \tilde{B}^{I}_{5,n+1} + \tilde{B}^{II}_{5,n+1} + \tilde{B}^{III}_{5,n+1} + \tilde{B}^{IV}_{5,n+1} + \tilde{B}^{V}_{5,n+1}$$

where

$$\begin{aligned}
\tilde{B}^{I}_{5,n+1} &= \tilde{B}^{I}_{5,n} + \tilde{B}^{II}_{5,n} + \tilde{B}^{III}_{5,n} \\
\tilde{B}^{II}_{5,n+1} &= \tilde{B}^{I}_{5,n} + \tilde{B}^{III}_{5,n} \\
\tilde{B}^{III}_{5,n+1} &= \tilde{B}^{V}_{5,n} \\
\tilde{B}^{IV}_{5,n+1} &= \tilde{B}^{II}_{5,n} \\
\tilde{B}^{V}_{5,n+1} &= \tilde{B}^{II}_{5,n}
\end{aligned} \qquad (10)$$

and $\tilde{B}^{I}_{5,5} = 14,\ \tilde{B}^{II}_{5,5} = 10,\ \tilde{B}^{III}_{5,5} = 2,\ \tilde{B}^{IV}_{5,5} = \tilde{B}^{V}_{5,5} = 4.$

<u>Proof</u>:
Extensions can be made by using either 2-by-2 or 3-by-3 tiles. A 2-by-2 tile will replace two adjacent 1-by-1 tiles in the last column. A 3-by-3 tile will replace a 2-by-2 tile and two 1-by-1 tiles that are adjacent to the 2-by-2 tile. After the interlocking tile has been placed, the remainder of the new last column is filled with 1-by-1 tiles. This may result in a set of four 1-by-1 tiles, which can be replaced by a 2-by-2 tile to form another basic block. We will show the possible extensions only for half of the configurations in Figure 17, as the remaining ones are symmetric (vertically reflected).

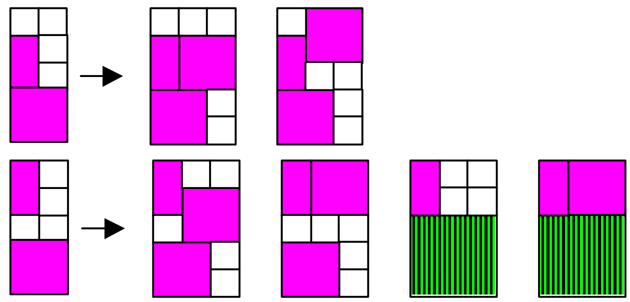

Figure 18

Type I: There are two ways to extend type I blocks with a 2-by-2 tile, leading to either a type I or a type II block of the next larger size. For the second type I block, two extensions with a 3-by-3 tile are also possible as shown in Figure 18.

Type II: In this case, an extension with a 2-by-2 tile is possible in only one way, creating a block of type I. In addition, an extension with a 3-by-3 tile is possible, just like in the case of the second basic block of type I, creating one block each of type IV and V. However, if we look at these two extensions in Figure 19 and compare them to those in Figure 18, we see that they create identical columns. The differences that existed before in the basic blocks of smaller size have now been erased. To avoid double counting, we will think of these extensions as coming from the type II blocks (then the two type I blocks create the same extension types).

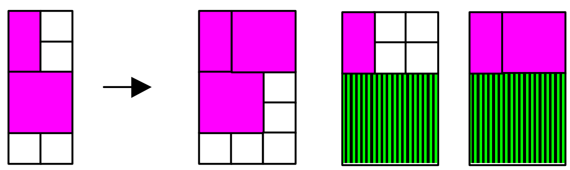

Figure 19

Thus, we can summarize what happens for type I and type II basic blocks:

Each type I block produces one type I and one type II block.
Each type II block produces one each of type I, type IV and type V.

Type III: A type III block can only be extended with a 2-by-2 tile, and there are two possible ways to do so, as shown in Figure 20. No extensions with 3-by-3 tiles are possible.

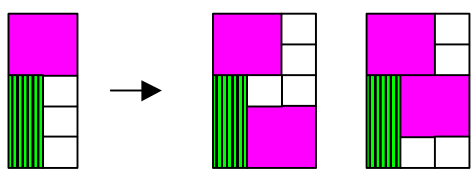

Figure 20

Type IV: This type has no extension.

Type V: There is only one way to extend a type V block, resulting in a type III block as shown in Figure 21.

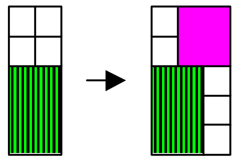

Figure 21

We summarize these three cases:

> Each type III block produces one type I and one type II block.
> A type IV block does not produce any extension.
> Each type V block produces one block of type III.

Thus, type I basic blocks of size 5-by-(n+1) are created by each of the type I, type II, and type III basic blocks of size 5-by-n. This results in:

$$\tilde{B}^{I}_{5,n+1} = \tilde{B}^{I}_{5,n} + \tilde{B}^{II}_{5,n} + \tilde{B}^{III}_{5,n}$$

Likewise, one derives the other four equations. As any extensible basic block of size 5-by-(n+1) has to be of one of these types for $n \geq 5$, the total number of basic blocks of size 5-by-(n+1) is the sum of the basic blocks of the types I - V.

Finally, we need to verify the initial conditions. Figure 16 shows the basic blocks containing 3-by-3 tiles. We only need to look at the basic blocks in the top row, as the bottom row contains basic blocks that are vertically reflected. Recall also that all but the middle basic block have two possibilities for tiling the (white) 2-by-2 area. The first basic block is of type II in both cases, the second is of type I in both cases, the third is of type III, and the fourth and fifth are either of type IV or type V. For the 18 possible basic blocks indicated in Figure 16, we get (as we double the above count): Type I: 4, type II: 4, type III: 2, type IV: 4, and type V: 4. Now we have to look at basic blocks of size 5-by-5 containing only 1-by-1 and 2-by-2 tiles. By Theorem 3, there are 16 such basic blocks. Figure 22 shows 8 of these, with the remaining 8 being symmetrical (vertical reflection).

The first, second, and seventh are of type II; the other ones are of type I. Thus, we have 6 additional blocks of type II and 10 additional basic blocks of type I. Altogether, this leads to $\tilde{B}^{I}_{5,5} = 14$, $\tilde{B}^{II}_{5,5} = 10$, $\tilde{B}^{III}_{5,5} = 2$, $\tilde{B}^{IV}_{5,5} = 4$, and $\tilde{B}^{V}_{5,5} = 4$. (This gives 34 of the 35 basic blocks of size 5-by-5; the remaining one is the basic block consisting of the 5-by-5 tile itself.)

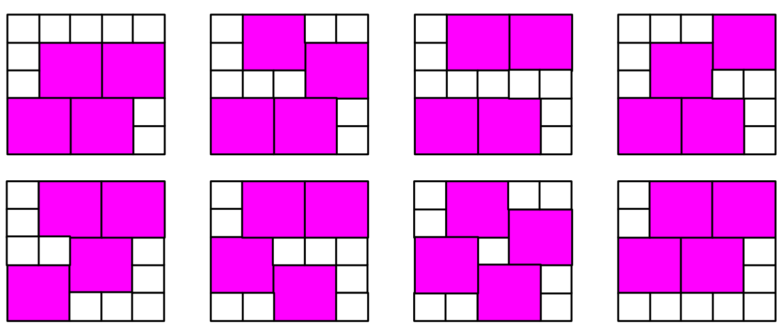

Figure 22

Finally, we can compute the number of tilings using the general recursive formula:

Theorem 6: *The number of tilings of a 5-by-n rectangle with squares of size up to 5-by-5 is given by*

$$\tilde{T}_{5,n} = \sum_{k=1}^{n} \tilde{B}_{5,k} \tilde{T}_{5,n-k} \ ,$$

where

$$\tilde{B}_{5,k} = \tilde{B}_{5,k-1} + \tilde{B}_{5,k-2} + \tilde{B}_{5,k-3} \quad \text{for } n > 8$$

with $\tilde{B}_{5,1} = 1, \tilde{B}_{5,2} = 7,\ \tilde{B}_{5,3} = 13,\ \tilde{B}_{5,4} = 20,\ \tilde{B}_{5,5} = 35,\ \tilde{B}_{5,6} = 66,\ \ \tilde{B}_{5,7} = 118,\ \tilde{B}_{5,8} = 218$, and $\tilde{T}_{5,0} = \tilde{T}_{5,1} = 1$.

Proof:
Using (10) of Lemma 6, we can express the number of basic blocks of type I and III in terms of the number of basic blocks of type II:

$$\tilde{B}^{I}_{5,n+1} = \tilde{B}^{II}_{5,n+1} + \tilde{B}^{II}_{5,n}, \text{ and } \tilde{B}^{III}_{5,n+1} = \tilde{B}^{II}_{5,n-1}. \qquad (11)$$

Summing over all types,

$$\begin{aligned}\tilde{B}_{5,n+1} &= \left(\tilde{B}^{II}_{5,n+1} + \tilde{B}^{II}_{5,n}\right) + \tilde{B}^{II}_{5,n+1} + \tilde{B}^{II}_{5,n-1} + \tilde{B}^{II}_{5,n} + \tilde{B}^{II}_{5,n} \\ &= 2\cdot\tilde{B}^{II}_{5,n+1} + 3\cdot\tilde{B}^{II}_{5,n} + \tilde{B}^{II}_{5,n-1}\end{aligned} \qquad (12)$$

Using (10) in combination with (11), we can derive a recursive formula for $\tilde{B}^{II}_{5,n+1}$:

$$\tilde{B}^{II}_{5,n+1} = \tilde{B}^{I}_{5,n} + \tilde{B}^{III}_{5,n} = \left(\tilde{B}^{II}_{5,n} + \tilde{B}^{II}_{5,n-1}\right) + \tilde{B}^{II}_{5,n-2}. \qquad (13)$$

Substituting (13) into (12) for each of the terms, followed by a suitable grouping of the resulting terms leads to the recursive equation for $\tilde{B}_{5,n+1}$:

$$\begin{aligned}\tilde{B}_{5,n+1} &= 2\cdot\left(\tilde{B}^{II}_{5,n} + \tilde{B}^{II}_{5,n-1} + \tilde{B}^{II}_{5,n-2}\right) + 3\cdot\left(\tilde{B}^{II}_{5,n-1} + \tilde{B}^{II}_{5,n-2} + \tilde{B}^{II}_{5,n-3}\right) \\ &\quad + \left(\tilde{B}^{II}_{5,n-2} + \tilde{B}^{II}_{5,n-3} + \tilde{B}^{II}_{5,n-4}\right) \\ &= \left(2\cdot\tilde{B}^{II}_{5,n} + 3\cdot\tilde{B}^{II}_{5,n-1} + \tilde{B}^{II}_{5,n-2}\right) + \left(2\cdot\tilde{B}^{II}_{5,n-1} + 3\cdot\tilde{B}^{II}_{5,n-2} + \tilde{B}^{II}_{5,n-3}\right) \\ &\quad + \left(2\cdot\tilde{B}^{II}_{5,n-2} + 3\cdot\tilde{B}^{II}_{5,n-3} + \tilde{B}^{II}_{5,n-4}\right) \\ &= \tilde{B}_{5,n} + \tilde{B}_{5,n-1} + \tilde{B}_{5,n-2}.\end{aligned}$$

This formula is valid for $n > 8$, since the recursions for the subtypes are only valid for $n > 5$. For $n \le 5$, the initial conditions for $\tilde{B}_{5,n}$ follow from Lemma 4. Equation (10) can be used together with the initial values given in Lemma 6 to compute the number of subtypes for $n = 6$, 7 and 8. Summing over all subtypes gives

$$\tilde{B}_{5,6} = 26+16+4+10+10 = 66,\ \tilde{B}_{5,7} = 46+30+10+16+16 = 118 \text{ and } \tilde{B}_{5,8} = 86+56+16+30+30 = 218.$$

$\boxed{\wedge}$

Table 6 shows the values of $\tilde{B}_{5,n}$ and $\tilde{T}_{5,n}$ for $n \le 10$.

n	1	2	3	4	5	6	7	8	9	10
$\tilde{B}_{5,n}$	1	7	13	20	35	66	118	218	402	738
$\tilde{T}_{5,n}$	1	8	28	117	472	1,916	7,765	31,497	127,707	517,881

Table 6

8. Conclusion

The approach used in this paper for generating basic blocks in the case where tiles of size up to k-by-k are allowed becomes quite complex as m increases (since more and more columns need to be taken into account for determining the different types). Therefore, results for $m > 5$ will most likely require a different approach.

However, in the case where only 1-by-1 and 2-by-2 tiles are used, the extension of basic blocks follows a more regular pattern. There is a good chance that combinatorial formulas for the number of basic blocks, similar to the one for $B_{m,2}$, may be derived. A first step is the implementation of an algorithm for generating and counting the basic blocks of the next larger size and to look for patterns in the resulting sequences.

Acknowledgement

I would like to thank Phyllis Chinn, who introduced me to this research question at a PROMPT (Professors Rethinking Options for Mathematics for Pre-service Teachers) workshop, sponsored by NSF grant TPE 92-53321. Thanks also to Daphne Liu, who was always willing to read new drafts of this paper. Finally, I would like to thank Neil Calkin, who has introduced me to the online check for integer sequences [7] and given me ideas for future work on this problem.

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