NEW RESULTS IN CIRCULAR NIM

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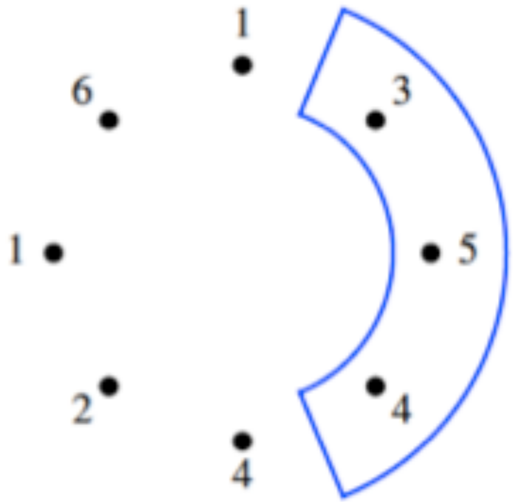
Definition of the Game CN(n,k)

- n stacks of tokens arranged in a circle
- Select k **consecutive** stacks and remove at least one token from at least one of the k stacks
- Last player to move wins
- $k = 1$ corresponds to NIM
- Example: CN(8,3)



Main Question:

Who wins in a combinatorial game from a specific position, assuming both players play optimally?

Impartial Games

Only two possible outcome classes:

- **Losing** positions (**P**-positions)
- **Winning** positions (**N**-positions)

Characterization of positions

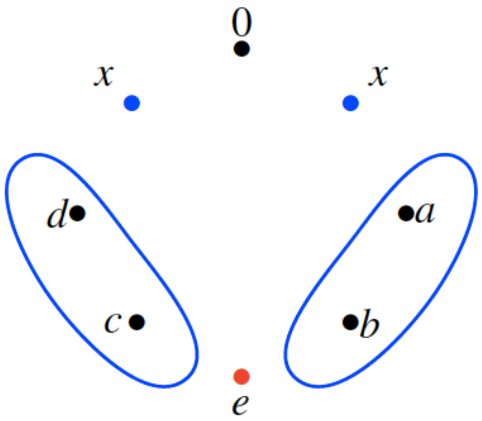
- From a **losing** position, **all** allowed moves lead to a **winning** position
- From a **winning** position, there is **at least one** move to a **losing** position.
- In normal play, the **terminal** positions are **losing** positions

Previous Results

- General results for CN(n,1), CN(n,n), and CN(n,n-1)
- These cover all the games for n ≤ 3
- Solved cases not covered by general results for n = 4, 5, and 6, with the exception of CN(6,2)
- Also proved result for CN(8,6)

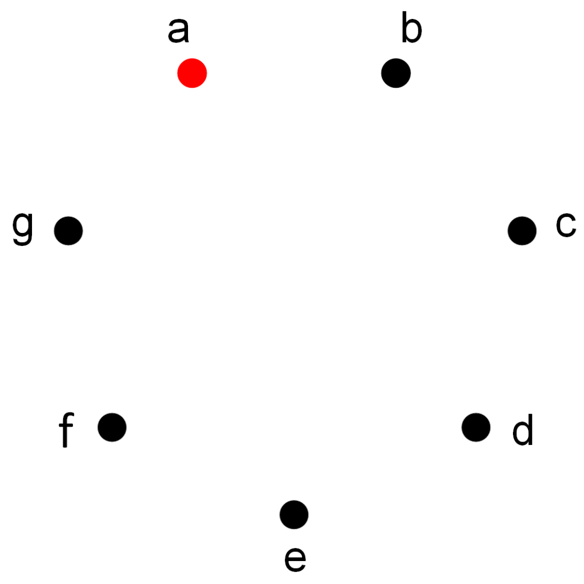
Theorem [DH]

The P-positions of CN(8,6) are given by {(0, x, a, b, e, c, d, x)| a + b = c + d = x, e = min { x, a + d }}



Results for n = 7 (new)

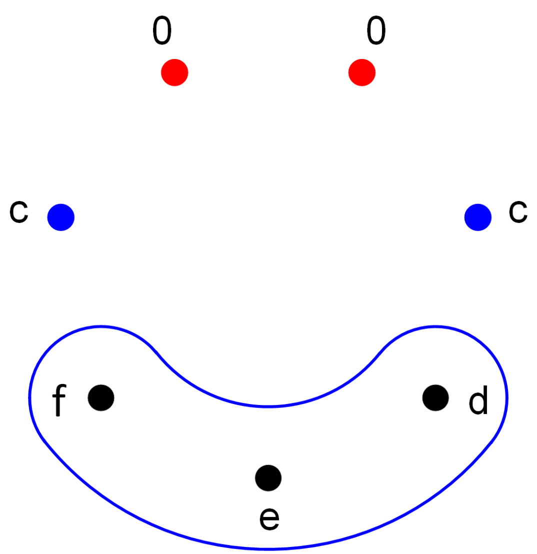
- We have results for CN(7,3) and CN(7,4)
- Generic position has *a* as the minimal value, and w.l.o.g., we assume that $b \leq g$



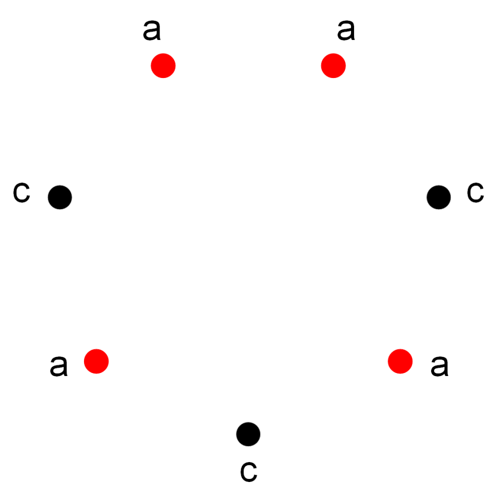
Results for CN(7,4)

Theorem Let a = min(**p**) and b ≤ g. The P-positions of CN(7,4) are given by one of the following 4 cases:

Case 1:
a = b = 0; c = g, d + e + f = c



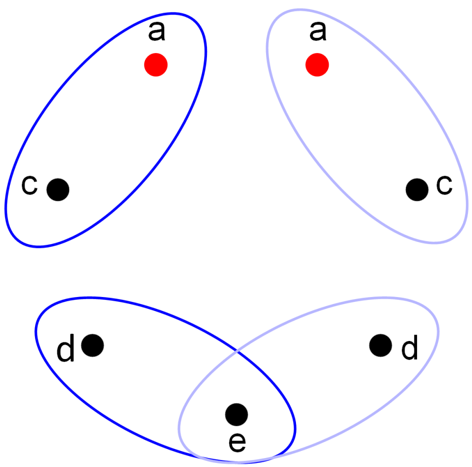
Case 2:
a = b = d = f; c = e = g



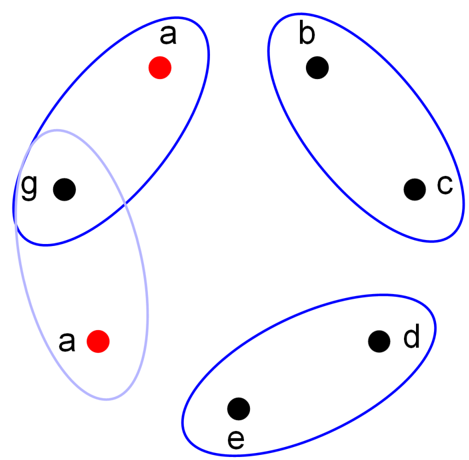
Results for CN(7,4)

Theorem Let a = min(**p**) and b ≤ g. The P-positions of CN(7,4) are given by one of the following 4 cases:

Case 3:
a = b, c = g, d = f
0 < a < min{c, d, e}
a + c = d + e



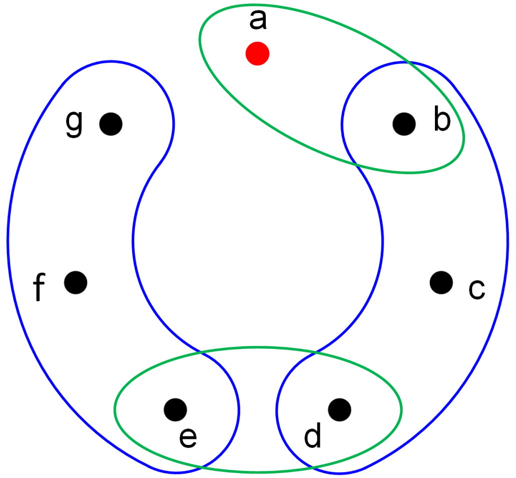
Case 4:
a = f
a < min{b, c, e, g}
b + c = d + e = g + a



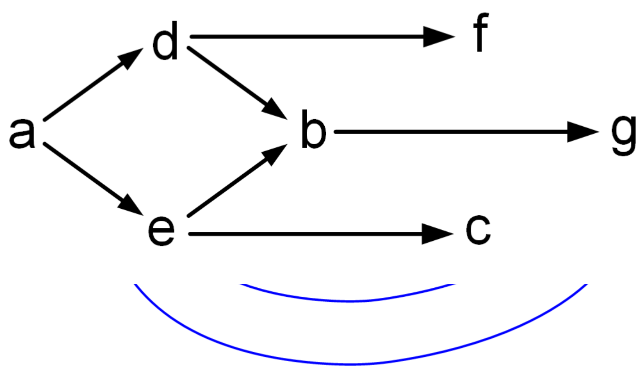
Results for CN(7,3)

Theorem Let a = min(**p**) and b ≤ g and **b + c + d = e + f + g**. The P-positions of CN(7,3) are given by one of the following 2 cases:

Case 1:
a + b = d + e



With the following inequalities

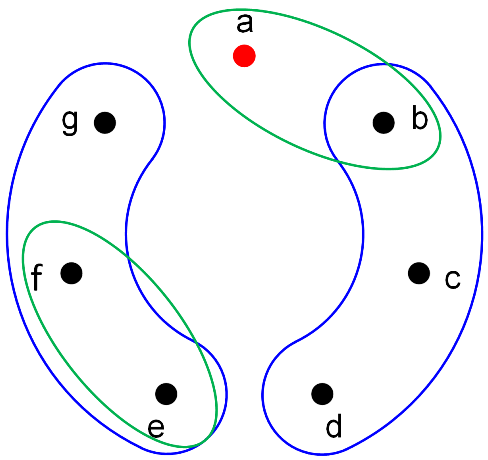


a ⟶ b indicates that a ≤ b

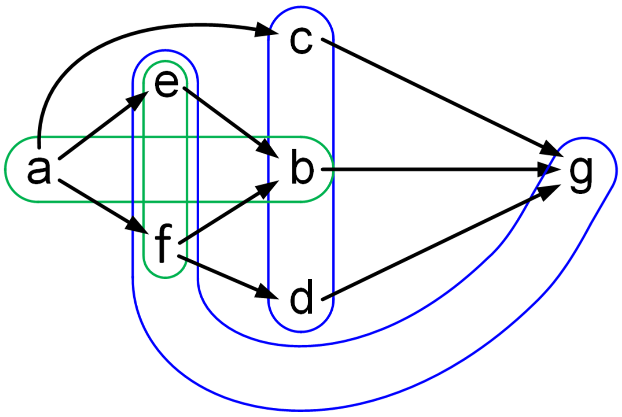
Results for CN(7,3)

Theorem Let a = min(**p**) and b ≤ g and **b + c + d = e + f + g**.
The P-positions of CN(7,3) are given by one of the following 2 cases:

Case 2:
a + b = e + f



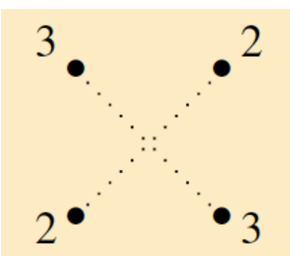
With the following inequalities



a ⟶ b indicates that a ≤ b

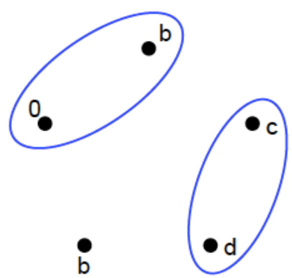
Open Questions

- Missing cases are CN(6,2), CN(7,2), CN(7,5)
- For n = 8, only CN(8,6) is known
- It seems that CN(n,2) are difficult to find
- Also, CN(n,n-2) do not show common structure

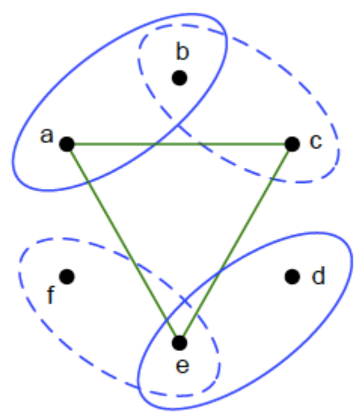


CN(4,2)

CN(5,3)

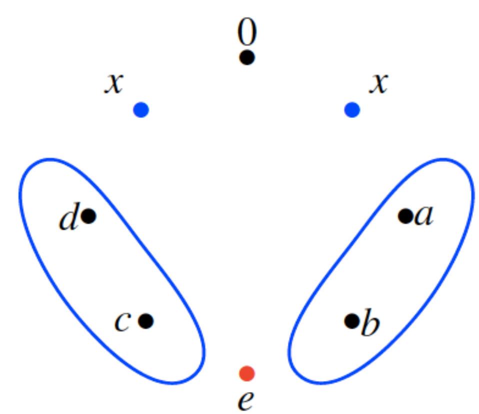


CN(6,4)



a = min(**p**)

CN(8,6)



CN(7,5) ?

THANKS!

Slides to be posted at

www.calstatela.edu/faculty/silvia-heubach

ANY QUESTIONS

Image from www.writewithexcellence.com