THE EXTENSIONAL STRUCTURE OF COMMUTATIVE NOETHERIAN RINGS

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Abstract. We show that finitely generated modules over a commutative Noetherian ring can be classified, up to isomorphism of submodule series, in a manner analogous to the classification of integers as products of prime numbers. In outline, two such modules have isomorphic submodule series if and only if 1) the set of minimal associated prime ideals of these modules coincide, 2) the multiplicities of these modules at these prime ideals coincide, and 3) the modules represent the same element in a certain group corresponding to the above set of prime ideals. Regarding condition 3), we show that, in the very special case that the ring is a Dedekind domain, the group corresponding to the prime ideal 0 is the ideal class group of the ring.

1. Introduction

Let R be a unital ring, R-**Mod** the category of left R-modules, and R-**Noeth** the category of Noetherian left R-modules. It is the hope of module theorists that the structure of these categories can be understood by considering the way in which a general module can be constructed from some small family of simpler modules. The prototype of this scheme is the classification of finitely generated Abelian groups, that is, $\mathbb{Z}$-**Noeth**. Here any such module is isomorphic to a finite direct sum of modules of the form $\mathbb{Z}/\mathbb{Z}p^n$ for $p, n \in \mathbb{N}$ with p prime, and of copies of $\mathbb{Z}$ itself.

A consequence of this classification (or perhaps part of its proof) is the existence of certain cancellation rules that hold in the category. For example, let $A, B, C, A_1, A_2, B_1, B_2 \in \mathbb{Z}$-**Noeth**, then

- *Cancellation:* If $A \oplus C \cong B \oplus C$ then $A \cong B$.
- *Multiplicative Cancellation:* If $A^n \cong B^n$ for some $n \in \mathbb{N}$, then $A \cong B$.

Much effort has been directed toward studying direct sum decompositions with the hope of generalizing this special case to more complicated rings. Unfortunately, almost no category of modules can be classified as easily as $\mathbb{Z}$-**Noeth**, and the failure of the above rules in, what one would think of as well behaved categories, such as R-**Noeth** when R is a commutative Noetherian ring, has been well documented [15]. This leaves the door open to the consideration of coarser classifications of modules – classification up to some other equivalence than isomorphism, or classification based on something other than direct sum decomposition.

In this paper, we will investigate a classification of modules in which the direct sum operation is replaced by extensions. Thus we consider a module B to be "composed" of modules A and C if there is a short exact sequence $0 \to A \to B \to C \to 0$. More generally, we consider two modules A and B to be equivalent if they can be constructed from the same set of modules using extensions. Specifically, A and B are equivalent, denoted $A \sim B$, if they have **isomorphic submodule series**, that is, there are series $0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$ and $0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n = B$ and a permutation of the indices σ, such that $A_i/A_{i-1} \cong B_{\sigma(i)}/B_{\sigma(i)-1}$ for $i = 1, 2, \ldots, n$.

Of course, knowing that $A \sim B$ is much weaker than knowing that $A \cong B$. But what we lose on the coarseness of the classification we gain in cancellation rules. Some hint of this can be seen in the following theorem:

Theorem 1.1. [4, 5.5] *Suppose R is a ring and $A, B, C \in R$-**Mod**. If $A \oplus C \cong B \oplus C$ with $C \in R$-**Noeth**, then A and B have isomorphic submodule series.*

The thing to notice here is that we make no hypothesis whatsoever on the ring R or the modules A and B.

The goal of this paper is to show that, when R is a commutative Noetherian ring, a partial classification of Noetherian modules up to isomorphism of submodule series is possible, and that this classification has many parallels with the classification of $\mathbb{Z}$-**Noeth** described above.

The natural way to prove this is to record the information about these equivalence classes in a commutative monoid: The class of $\sim$-equivalence classes, together with the operation induced on these equivalence classes by the direct sum, forms a commutative monoid, which we will call $M(R\text{-}\mathbf{Mod})$ ([4, 3.5]). We write $[A] \in M(R\text{-}\mathbf{Mod})$ for the $\sim$-equivalence class of modules which contains $A \in R$-**Mod**. The image of R-**Noeth** in $M(R\text{-}\mathbf{Mod})$ is a submonoid which we will call $M(R\text{-}\mathbf{Noeth})$. We note that any monoid has a preorder $\leq$ defined by $a \leq b$ if there is some c such that $a + c = b$. In the special case of $M(R\text{-}\mathbf{Mod})$ and $M(R\text{-}\mathbf{Noeth})$ we have $[A] \leq [B]$ if and only if there are submodule series $0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$ and $0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_m = B$ and an injective map σ, such that $A_i/A_{i-1} \cong B_{\sigma(i)}/B_{\sigma(i)-1}$ for $i = 1, 2, \ldots, n$ (3.2(2)).

Classifying Noetherian modules up to isomorphism of submodule series is then the same as understanding the structure of $M(R\text{-}\mathbf{Noeth})$. The main tools here are the cancellation rules which hold in $M(R\text{-}\mathbf{Noeth})$, together with certain homomorphisms on $M(R\text{-}\mathbf{Noeth})$ and certain embedded Abelian groups:

- There is a surjection, called radann, from $M(R\text{-}\mathbf{Noeth})$ to the set of radical ideals of R defined by $\operatorname{radann}([A]) = \operatorname{rad}(\operatorname{ann} A)$ for all $A \in R\text{-}\mathbf{Noeth}$ (Section 4). Thus for a module $A \in R\text{-}\mathbf{Noeth}$, the prime ideals which are minimal over $\operatorname{ann} A$ are determined by $[A]$.
- For each prime ideal P of R, there is a monoid homomorphism $\phi_{[R/P]}$ on $M(R\text{-}\mathbf{Noeth})$ which records a "multiplicity" with respect to P. Specifically, for $A \in R\text{-}\mathbf{Noeth}$, $\phi_{[R/P]}([A])$ counts the number of times R/P can appear as a factor in a submodule series for A (3.7).
- For each element $[A]$ of the monoid there is an associated Abelian group $G_{[A]}$ (2.6) which embeds in $M(R\text{-}\mathbf{Noeth})$ as the set of elements $[B] \in M(R\text{-}\mathbf{Noeth})$ such that $[A] \leq [B] \leq [A]$.

With these tools at hand, we can now describe the classification of Noetherian modules up to isomorphism of submodule series:

Theorem 1.2. (5.4) *Suppose R is a commutative Noetherian ring, and $A, B \in R\text{-}\mathbf{Noeth}$. Then A and B have isomorphic submodule series (equivalently, $A \sim B$ or $[A] = [B]$) if and only if*

(1) $\operatorname{radann}[A] = \operatorname{radann}[B]$. *This condition is equivalent to the set of prime ideals $\{P_1, P_2, \dots, P_n\}$ which are minimal over* $\operatorname{ann} A$ *being the same as the set of primes which are minimal over* $\operatorname{ann} B$. *Either condition implies that $G_{[A]} = G_{[B]}$.*
(2) $\phi_{[R/P_i]}([A]) = \phi_{[R/P_i]}([B])$ *for $i = 1, 2, \dots, n$ with $\{P_1, P_2, \dots, P_n\}$ as in (1). Conditions (1) and (2) imply that $[A] \leq [B] \leq [A]$.*
(3) *A and B represent the same element in the group $G_{[A]} = G_{[B]}$.*

This theorem is not so foreign as it seems — all of its features already occur in the classification of integers as products of prime numbers: Every nonzero integer $N \in \mathbb{Z}$ can be written in the form $N = up_1^{n_1}p_2^{n_2}\dots p_n^{n_n}$ where $\{p_1, p_2, \dots, p_n\}$ are distinct (positive) prime numbers, n_i is the "multiplicity" of the prime p_i in N, and u is an element of the group of units $\{+1, -1\}$ of $\mathbb{Z}$. Two integers are equal if and only if the corresponding sets of prime numbers, multiplicities and group elements match up almost as described in the theorem. (To make this parallel precise one has only to notice that $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ with multiplication as operation is a primely generated refinement monoid just as $M(R\text{-}\mathbf{Noeth})$ is, and so the theory in Section 2 applies to both.)

As mentioned above, the proof of Theorem 1.2 depends very much on the cancellation properties of $M(R\text{-}\mathbf{Noeth})$: Though $M(R\text{-}\mathbf{Noeth})$ is seldom cancellative, it is always strongly separative (2.3, 3.9), meaning that, for any $A, B, C \in R\text{-}\mathbf{Noeth}$, $[A] + [C] = [B] + [C]$ and $[C] \leq [A]$ imply $[A] = [B]$. And though $M(R\text{-}\mathbf{Noeth})$ does not have multiplicative cancellation exactly analogous to the property of $\mathbb{Z}\text{-}\mathbf{Noeth}$ described above, it does satisfy (3.9, 2.13(P5)) $n[A] \leq n[B]$ implies $[A] \leq [B]$ for any $A, B \in R\text{-}\mathbf{Noeth}$ and $n \in \mathbb{N}$. Many other properties of $M(R\text{-}\mathbf{Noeth})$ appear in 3.9 and 2.13.

Of course, Theorem 1.2 is not really a complete classification unless it is known what $G_{[A]}$ is for each $A \in R\text{-}\mathbf{Noeth}$. It follows from 1.2(1) that $G_{[A]} = G_{[R/S]}$ where $S = \operatorname{rad}(\operatorname{ann} A)$, so there is really only one such group for each radical ideal of R. In the special case that R is a Dedekind domain we will show (6.3) that the group $G_{[R]}$ is isomorphic to the ideal class group of the ring. So there is some important nontrivial structure recorded in these groups and hence in $M(R\text{-}\mathbf{Noeth})$. This also means that, in general, calculating $G_{[A]}$ will be at least as difficult as calculating the ideal class groups of a Dedekind domain.

2. Primely Generated Refinement Monoids

All monoids in this paper will be commutative, so we will write $+$ for the monoid operation and 0 for the identity element of all monoids. We refer the reader to [14] and [13] for the standard concepts of monoid theory.

We collect here some notation we will need:

Notation 2.1. *Let M be a monoid and $a, b \in M$.*

- $a \leq b \iff \exists\, c \in M$ *such that* $a + c = b$
- $a \ll b \iff a + b \leq b$
- $a \equiv b \iff a \leq b$ *and* $b \leq a$
- $\{\equiv a\} = \{\, c \in M \mid c \equiv a \,\}$
- $a \propto b \iff \exists\, n \in \mathbb{N}$ *such that* $a \leq nb$
- $\{\propto a\} = \{\, c \in M \mid c \propto a \,\}$
- $a \asymp b \iff b \propto a \propto b$
- $\{\asymp a\} = \{\, c \in M \mid c \asymp a \,\}$

The relation $\leq$ is a preorder on M. For the monoid $(\mathbb{Z}^+, +)$, the set of nonnegative integers, this preorder coincides with the usual order. If M is a group, then we have $a \leq b$ for all $a, b \in M$. So for the monoid $(\mathbb{Z}, +)$, the preorder $\leq$ is not the same as the usual order on the integers.

The relation $\ll$ is transitive, $\propto$ is a preorder, and $\equiv$ and $\asymp$ are congruences. These relations behave well under monoid homomorphisms: If

$\phi\colon M \to N$ is monoid homomorphism, and $a, b \in M$, then $a \leq b$ implies $\phi(a) \leq \phi(b)$, $a \ll b$ implies $\phi(a) \ll \phi(b)$, etc.

Let I be a submonoid of a monoid M and $x, y \in I$. It is trivial that if $x \leq y$ with respect to the preorder in I, then $x \leq y$ with respect to the preorder in M. The converse may not be true: Consider, for example, the submonoid $\mathbb{Z}^+ \subseteq \mathbb{Z}$. Thus we will distinguish certain submonoids which behave well with respect to the preorder $\leq$:

Definition 2.2. *An* **order ideal** [4, 2.1] *of a monoid M is a submonoid $I \subseteq M$ such that*

$$(\forall x, y \in M)\ (x \leq y \in I \implies x \in I).$$

Here $\leq$ is the preorder defined with respect to M.

The following facts are easy to check:

- $I \subseteq M$ is an order ideal if and only if
$$(\forall x, y \in M)\ (x + y \in I \iff x, y \in I).$$
- If I is an order ideal of a monoid M, then $x \leq y$ in I if and only if $x \leq y$ in M.
- The intersection of a family of order ideals is again an order ideal.
- The order ideal generated by an element $a \in M$ is $\{\propto a\}$.

Among the many cancellation properties that can occur in monoids, the most important, after cancellation itself, are separativity and strong separativity:

Definition 2.3. [1], [3], [4] *A monoid M is* **separative** *if it satisfies any of the following equivalent conditions:*

S1. $(\forall a, b \in M)\ (2a = a + b = 2b \implies a = b)$
S2. $(\forall a, b, c \in M)\ ((a + c = b + c \textit{ and } c \propto a, b) \implies a = b)$
S3. $(\forall a \in M)\ (\{\asymp a\} \textit{ is cancellative})$
S4. $(\forall a, b \in M)(\forall m, n \in \mathbb{N})\ ((ma = mb \textit{ and } na = nb)$
$\implies (ka = kb \textit{ where } k = \gcd(m, n)))$

A monoid M is **strongly separative** *if it satisfies any of the following equivalent conditions:*

T1. $(\forall a, b \in M)\ (2a = a + b \implies a = b)$
T2. $(\forall a, b, c \in M)\ ((a + c = b + c \textit{ and } c \leq a) \implies a = b)$
T3. $(\forall a, b, c \in M)\ ((a + c = b + c \textit{ and } c \propto a) \implies a = b)$
T4. $(\forall a, b \in M)(\forall n \in \mathbb{N})\ ((n + 1)a = na + b \implies a = b)$

It is obvious from S1 and T1 that strong separativity implies separativity for any monoid.

Definition 2.4. *Let M be a monoid and $a \in M$.*

1. *a is* **regular** *if $2a \leq a$ (equivalently, $2a \equiv a$ or $a \ll a$).*
2. *a is* **free** *if for all $m, n \in \mathbb{N}$, $ma \leq na$ implies $m \leq n$.*

Evidently, an element of a monoid cannot be both free and regular, but it is possible for an element to be neither. For example, in the monoid $M = \{0, 1, \infty\}$, where $1 + 1 = 1 + \infty = \infty + \infty = \infty$, the element 1 is neither free nor regular. Notice that, by construction, this monoid is not separative.

Proposition 2.5. *Let a be an element of a monoid M.*

1. *If M is separative then a is free or regular.*
2. *If M is strongly separative then a is free or $a \leq 0$.*

Proof.

1. If M is separative and a is not free, then there are $m, n \in \mathbb{N}$ such that $m > n$ and $ma \leq na$. In particular, $(n+1)a + x = na$ for some $x \in M$. Thus we have $(2a + x) + (n-1)a = a + (n-1)a$ with $(n-1)a \propto a \leq 2a + x$. Using 2.3(S2), we get $2a + x = a$ and hence $2a \leq a$, that is, a is regular.
2. If M is strongly separative, then it is separative and a is either free or regular. But if $2a \leq a$, then $(a + x) + a = a$ for some $x \in M$, with $a \propto a + x$. Using 2.3(T3), we get $a + x = 0$. In particular, $a \leq 0$. □

Associated with any element u of any monoid is an Abelian group G_u which we construct as follows:

Definition 2.6. *Let u be an element of a monoid M. Define a congruence $\sim_u$ on M by*

$$a \sim_u b \iff u + a = u + b$$

for $a, b \in M$. We will write $[a]_u$ for the $\sim_u$-congruence class containing $a \in M$. Define $G_u = \{\, [a]_u \mid a \ll u \,\}$. One can easily show that G_u is the set of all units (invertible elements) of the quotient monoid $M/\!\sim_u$ and so is an Abelian group.

The following facts about G_u are easy to check:

Lemma 2.7. *Let u, v, w be elements of a monoid M.*

1. *If $v \equiv u$, then $\sim_v$ and $\sim_u$ coincide and hence $G_u = G_v$.*
2. *If M is separative and $v \asymp u$, then $\sim_v$ and $\sim_u$ coincide and hence $G_u = G_v$.*

(3) *The map* $\Omega\colon G_u \to \{\equiv u\}$ *defined by* $\Omega([x]_u) = u + x$ *is a bijection (but not in general a homomorphism). If we define the operation* $\square_u$ *on* $\{\equiv u\}$ *by* $a \square_u b = u + x + y$ *where* $a = u + x$ *and* $b = u + y$, *then the set* $\{\equiv u\}$ *with operation* $\square_u$ *is a group isomorphic to* G_u, *with identity* u.

(4) *If* M *is strongly separative, the operation* $\square_u$ *can be expressed in a simpler way:* $a \square_u b = c$ *where* $a + b = u + c$.

(5) *If* $u \leq v$, *then the map* $\sigma_{uv}\colon G_u \to G_v$ *defined by* $\sigma_{vu}([a]_u) = [a]_v$ *is a group homomorphism. Further, if* $u \leq v \leq w$, *then* $\sigma_{wu} = \sigma_{wv} \circ \sigma_{vu}$.

From 3, we see that we could have defined G_u to be $(\{\equiv u\}, \square_u, u)$. The advantage of this is that the elements of the group are then elements of M, rather than congruence classes. The disadvantage is that if $v \equiv u$, then we have $G_u = G_v$ as sets, but the operations $\square_u$ and $\square_v$ are, in general, different.

Definition 2.8. *A monoid* M *has* **refinement** [21], [8], [9], [22], *if for all* $a_1, a_2, b_1, b_2 \in M$ *with* $a_1 + a_2 = b_1 + b_2$, *there exist* $c_{11}, c_{12}, c_{21}, c_{22} \in M$ *such that*

$$a_1 = c_{11} + c_{12} \qquad a_2 = c_{21} + c_{22}$$
$$b_1 = c_{11} + c_{21} \qquad b_2 = c_{12} + c_{22}.$$

Definition 2.9. *Let* M *be a monoid and* $a, p \in M$.

(1) p *is* **prime** *if for all* $a_1, a_2 \in M$, $p \leq a_1 + a_2$ *implies* $p \leq a_1$ *or* $p \leq a_2$.
(2) $a \in M$ *is* **primely generated** *if it is a sum of prime elements.*
(3) M *is* **primely generated** *if all its elements are primely generated.*

Notice that, by this definition, any element $p \leq 0$ is prime. Also, if $p \equiv q$, then p is prime if and only if q is prime. It is easy to check that, if $p \in M$ is prime and $p = a_1 + a_2$, then $p \equiv a_1$ or $p \equiv a_2$.

Lemma 2.10. *Let* I *be an order ideal of a refinement monoid* M. *Then* I *has refinement, and*

(1) *If* $p \in I$, *then* p *is a prime (free) element of* M *if and only if* p *is a prime (free) element of* I.
(2) *If* M *is primely generated, then so is* I.

For an example, consider the submonoid $I = \{0, 2, 3, 4, \dots\} = \mathbb{Z}^+ \setminus \{1\}$ of the refinement monoid $\mathbb{Z}^+$. The submonoid I is not an order ideal, and the elements 2 and 3 are prime in I, but not in $\mathbb{Z}^+$.

We collect in the next three lemmas some properties of primely generated refinement monoids which are proved in [6].

Lemma 2.11. [6, 5.4, 5.5] *Let p be a prime element of a refinement monoid M. If $p \not\leq 0$, there is a monoid homomorphism $\phi_p\colon M \to \mathbb{Z}^\infty$ defined by*

$$\phi_p(a) = \sup\{\, n \in \mathbb{Z}^+ \mid np \leq a \,\}$$

for all $a \in M$. Here $\mathbb{Z}^\infty = \mathbb{Z}^+ \cup \{\infty\}$ with $n + \infty = \infty$ for all $n \in \mathbb{Z}^+$ and $\infty + \infty = \infty$. Further, $\phi_p(p) = 1$ if and only if p is free, and $\phi_p(p) = \infty$ if and only if p is regular.

In the following lemma, a subset X of a monoid M is **incomparable** if $(x_1 \leq x_2 \implies x_1 = x_2)$ for all $x_1, x_2 \in M$.

Lemma 2.12. [6, 5.8, 5.9, 5.18] *Let a be a primely generated element in a refinement monoid M. Then a is either free or is regular.*

If a is free, then there is an incomparable set of (free) prime elements $\{p_1, p_2, \ldots, p_n\} \subseteq M$ such that

$$a \equiv \phi_{p_1}(a)p_1 + \phi_{p_2}(a)p_2 + \ldots + \phi_{p_n}(a)p_n,$$

with ϕ_{p_i} as in 2.11. The primes $p_1, p_2, \ldots, p_n$ are uniquely determined up to $\equiv$-congruence. Moreover, the map $\psi\colon \mathbb{N}^n \times G_a \to \{\asymp a\}$ defined by

$$\psi(n_1, n_2, \ldots, n_n, [x]_a) = n_1p_1 + n_2p_2 + \ldots + n_np_n + x$$

is a semigroup isomorphism.

Theorem 2.13. *Let M be a primely generated refinement monoid. Then M is separative. If in addition, for all primes p in the generating set, either p is free or $p \leq 0$, then M is strongly separative. In either case, M has the following properties:*

Pseudo-cancellation properties:

P1. $(\forall a, b, c \in M)\ (a + c \leq b + c \implies (\exists a_1 \ll c \text{ such that } a \leq b + a_1))$

P2. $(\forall a, c_1, c_2 \in M)\ (a \ll c_1 + c_2 \implies$
$\qquad (\exists a_1, a_2 \text{ such that } a = a_1 + a_2,\ a_1 \ll c_1 \text{ and } a_2 \ll c_2))$

Archimedean properties:

P3. $(\forall a, b \in M)\ ((\forall n \in \mathbb{N})(na \leq b) \implies a \ll b)$

P4. $(\forall a, b \in M)\ ((\forall n \in \mathbb{N})(na \leq (n+1)b) \implies a \leq b)$

Unperforation properties:

P5. $(\forall a, b \in M)(\forall n \in \mathbb{N})\ (na \leq nb \implies a \leq b)$

P6. $(\forall a, b \in M)(\forall n \in \mathbb{N})\ (na \equiv nb \implies a \equiv b)$

Join property:

P7. $(\forall c_1, c_2 \in M)\ (\exists d \text{ such that } c_1, c_2 \leq d \text{ and}$
$\qquad (\forall a \in M)\ (c_1, c_2 \leq a \implies d \leq a))$

The names *pseudo-cancellation*, and *unperforation* are from Wehrung [22], [23], [24]. P2 is a consequence of P1 in any refinement monoid.

Properties P3-P6 are trivially true of the monoid $\mathbb{Z}^\infty$ defined in 2.11. They are true also of M because, for each prime p in the generating set, 2.11 provides a homomorphism $\phi_p\colon M \to \mathbb{Z}^\infty$. See [6, Section 5] for details.

If M happened to be partially ordered by $\leq$, then property P7 would say that $(M, \leq)$ is a join-semilattice.

Corollary 2.14. *Let $u, u_1, u_2, \dots, u_n$ be elements of a primely generated refinement monoid M such that $u = u_1 + u_2 + \dots + u_n$. Then the map $\Gamma\colon G_{u_1} \oplus G_{u_2} \oplus \dots \oplus G_{u_n} \to G_u$ defined by*

$$\Gamma([a_1]_{u_1}, [a_2]_{u_2}, \dots, [a_n]_{u_n}) = [a_1 + a_2 + \dots + a_n]_u$$

is a surjective group homomorphism.

Proof. The map Γ is the sum of the homomorphisms σ_{uu_i} for $i = 1, 2, \dots, n$ as in 2.7(5), so is itself a homomorphism. If $[a]_u \in G_u$ then $a \ll u$. Using 2.13(P2) inductively we can find write $a = a_1 + a_2 + \dots + a_n$ with $a_i \ll u_i$ and hence $[a_i]_{u_i} \in G_{u_i}$ for all i. Thus $\Gamma([a_1]_{u_1}, [a_2]_{u_2}, \dots, [a_n]_{u_n}) = [a]_u$. $\square$

3. Extension Properties

Until further notice, R is any ring, and R-**Mod** the category of left R-modules. A **Serre subcategory** of R-**Mod** is a full subcategory $\mathbb{S}$ of R-**Mod** such that if $0 \to A \to B \to C \to 0$ is a short exact sequence in R-**Mod**, then $B \in \mathbb{S}$ if and only if $A, C \in \mathbb{S}$. The main example of a Serre subcategory is R-**Noeth**, the subcategory of all Noetherian left R-modules.

As described in [3], [4] and [5], associated with each Serre subcategory $\mathbb{S}$ is a refinement monoid $M(\mathbb{S})$ and a map $\Psi_{\mathbb{S}}\colon \mathbb{S} \to M(\mathbb{S})$ with the following universal property: If N is a monoid and $\Lambda\colon \mathbb{S} \to N$ satisfies the conditions

i) $\Lambda(0) = 0$,
ii) $\Lambda(B) = \Lambda(A) + \Lambda(C)$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathbb{S}$,

then there is a unique monoid homomorphism $\lambda\colon M(\mathbb{S}) \to N$ such that $\Lambda(A) = \lambda(\Psi_{\mathbb{S}}(A))$ for all $A \in \mathbb{S}$. Maps satisfying the conditions i and ii above are said to **respect short exact sequences** in $\mathbb{S}$.

We should point out that the monoid $M(\mathbb{S})$ may not be a set. In fact for any nontrivial ring R, $M(R\text{-}\mathbf{Mod})$ is always a proper class [3, 16.13]. We will see, for example, that if R is a field, then $M(R\text{-}\mathbf{Mod})$ is isomorphic to $(\mathbf{Card}, +, 0)$, the class of cardinal numbers with cardinal addition as operation. Though **Card** is not a set, it has the property that for any $\alpha \in \mathbf{Card}$, $\{\propto \alpha\}$ is a set. Similarly for any ring R and $A \in R$-**Mod**, the order ideal $\{\propto [A]\}$ is a set [3, 16.14]. Further, since the class of isomorphism classes in R-**Noeth** is a set, $M(R\text{-}\mathbf{Noeth})$ is a set for any ring R.

We will see shortly that, for any Serre subcategory $\mathbb{S}$, it is convenient to consider $M(\mathbb{S})$ to be contained in $M(R\text{-}\mathbf{Mod})$, thus we will focus first on the properties of $M(R\text{-}\mathbf{Mod})$. We will write $[A]$ rather than $\Psi_{R\text{-}\mathbf{Mod}}(A)$ for the image of a module $A \in R\text{-}\mathbf{Mod}$ in $M(R\text{-}\mathbf{Mod})$.

From [4, Section 3] and [5, Section 4], for $A, B \in R\text{-}\mathbf{Mod}$ we have that $[A] = [B]$ if and only if A and B have **isomorphic submodule series**, that is, there are submodule series $0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$ and $0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n = B$ and a permutation of the indices σ, such that $A_i/A_{i-1} \cong B_{\sigma(i)}/B_{\sigma(i)-1}$ for $i = 1, 2, \ldots, n$.

We will also need module level descriptions of the relations $[A] \leq [B]$, $[A] \leq [B_1] + [B_2]$, $[A_1] + [A_2] = [B_1] + [B_2]$ and others. We will avoid considerable notational difficulty in these descriptions by using the following definition:

Definition 3.1. [5, 4.1] *A* **partition** *is a finite indexed set of modules* $\mathbb{A} = (A_i)_{i\in\mathcal{I}}$.

Let $\mathbb{A} = (A_i)_{i\in\mathcal{I}}$ *and* $\mathbb{B} = (B_j)_{j\in\mathcal{J}}$ *be two partitions. Then* $\mathbb{A}$ *and* $\mathbb{B}$ *are* **isomorphic** *if there is a bijection* $\sigma\colon \mathcal{I} \to \mathcal{J}$ *such that* $A_i \cong B_{\sigma(i)}$ *for all* $i \in \mathcal{I}$, *and* $\mathbb{A}$ *is a* **subpartition** *of* $\mathbb{B}$ *if there is an injection* $\sigma\colon \mathcal{I} \to \mathcal{J}$ *such that* $A_i \cong B_{\sigma(i)}$ *for all* $i \in \mathcal{I}$.

A **partition of** $A \in R\text{-}\mathbf{Mod}$ *is a partition* $\mathbb{A} = (A_i)_{i\in\mathcal{I}}$ *such that there is a submodule series* $0 = A'_0 \subseteq A'_1 \subseteq \cdots \subseteq A'_n = A$ *and a bijection* $\sigma\colon \mathcal{I} \to \{1, 2, .., n\}$ *with* $A_i \cong A'_{\sigma(i)}/A'_{\sigma(i)-1}$ *for all* $i \in \mathcal{I}$. *A* **subpartition of** $A \in R\text{-}\mathbf{Mod}$ *is a subpartition of a partition of* A

A partition $\mathbb{B} = (B_j)_{j\in\mathcal{J}}$ *is a* **refinement** *of partition* $\mathbb{A} = (A_i)_{i\in\mathcal{I}}$ *if* $\mathcal{J}$ *can be written as a disjoint union of subsets* $(\mathcal{J}_i)_{i\in\mathcal{I}}$ *such that for all* $i \in \mathcal{I}$, $(B_j)_{j\in\mathcal{J}_i}$ *is a partition of* A_i. *Note that if* $\mathbb{A}$ *is a partition of* $A \in R\text{-}\mathbf{Mod}$, *then any refinement of* $\mathbb{A}$ *is also a partition of* A. *Further, if* $\mathbb{A}$ *and* $\mathbb{B}$ *are isomorphic partitions, then any refinement of* $\mathbb{A}$ *is induces an isomorphic refinement of* $\mathbb{B}$ *and vice versa.*

Using these definitions, for $A, B \in R\text{-}\mathbf{Mod}$ we have $[A] = [B]$ if and only if A and B have isomorphic partitions. The Schreier refinement theorem [10, 3.10] says that if $\mathbb{A}$ and $\mathbb{A}'$ are two partitions of a module $A \in R\text{-}\mathbf{Mod}$, then $\mathbb{A}$ and $\mathbb{A}'$ have isomorphic refinements. It follows easily from this theorem that $M(R\text{-}\mathbf{Mod})$ is a refinement monoid [4, 3.8].

Lemma 3.2. [5, 4.3] *Let* $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m \in R\text{-}\mathbf{Mod}$.

(1) $[A_1] + [A_2] + \ldots + [A_n] = [B_1] + [B_2] + \ldots + [B_m]$ *if and only if the partitions* $(A_1, A_2, \ldots, A_n)$ *and* $(B_1, B_2, \ldots, B_m)$ *have isomorphic refinements.*

(2) $[A_1] + [A_2] + \ldots + [A_n] \leq [B_1] + [B_2] + \ldots + [B_m]$ *if and only if* $(A_1, A_2, \ldots, A_n)$ *has a refinement which is a subpartition of a refinement of* $(B_1, B_2, \ldots, B_m)$.

For example, for $A, B \in R$-**Mod**, we have from this lemma that $[A] \leq [B]$ if and only if A has a partition which is a subpartition of B.

If $0 \to A \to B \to C \to 0$ is a short exact sequence in R-**Mod**, then (A, C) is a partition of B, and so from 3.2(1), $[B] = [A] + [C]$. Thus the map $A \mapsto [A]$ respects short exact sequences. Some other simple properties are collected in the next lemma.

Lemma 3.3. *Let R be a ring and $A, B \in R$-**Mod**.*

(1) $A = 0 \iff [A] \leq 0 \iff [A] = 0$
(2) $A \cong B \implies [A] = [B]$
(3) $[A \oplus B] = [A] + [B]$
(4) *If A is isomorphic to a submodule, factor module or subfactor module of B, then* $[A] \leq [B]$
(5) *If $\mathbb{A} = (A_i)_{i \in \mathcal{I}}$ is a partition of A, then* $[A] = \sum_{i \in \mathcal{I}} [A_i]$.
(6) *If A is finitely generated, then* $[A] \propto [R]$.

We can now specify how $M(\mathbb{S})$ embeds in $M(R\text{-}\mathbf{Mod})$ for a Serre subcategory $\mathbb{S}$:

Lemma 3.4. [3, 16.8][4, 3.9] *Let R be a ring.*

(1) *Given an order ideal I of $M(R\text{-}\mathbf{Mod})$, the class*

$$\mathbb{S} = \{ A \in R\text{-}\mathbf{Mod} \mid [A] \in I \}$$

*is a Serre subcategory of R-**Mod**. Moreover $I \cong M(\mathbb{S})$.*

(2) *Given a Serre subcategory $\mathbb{S}$ of R-**Mod**, the class*

$$I = \{ [A] \in M(R\text{-}\mathbf{Mod}) \mid A \in \mathbb{S} \}$$

is an order ideal of $M(R\text{-}\mathbf{Mod})$. Moreover $I \cong M(\mathbb{S})$.

From this lemma we see that there is a bijection between Serre subcategories of R-**Mod** and order ideals of $M(R\text{-}\mathbf{Mod})$.

In view of this lemma we will make the convention that, for any Serre subcategory $\mathbb{S}$ of R-**Mod**, $M(\mathbb{S})$ is contained (as an order ideal) in $M(R\text{-}\mathbf{Mod})$. In particular, $M(R\text{-}\mathbf{Noeth}) \subseteq M(R\text{-}\mathbf{Mod})$. Then $[A]$ is an element of $M(\mathbb{S})$ or $M(R\text{-}\mathbf{Noeth})$ according to whether A is in $\mathbb{S}$ or R-**Noeth** respectively.

Now suppose we have a functor $\Gamma\colon S\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ where S is a ring. If Γ is exact, it is easy to see that Γ induces a monoid homomorphism $\gamma\colon M(S\text{-}\mathbf{Mod}) \to M(R\text{-}\mathbf{Mod})$ such that $\gamma([A]) = [\Gamma(A)]$ for all $A \in S\text{-}\mathbf{Mod}$.

An easy example of such a functor is obtained from a ring homomorphism $\phi\colon R \to S$. Let $\Gamma\colon S\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ be the functor which sends an S-module ${}_SA$ to ${}_RA$, the same module with R action determined by ϕ, and maps an S-module homomorphism to the same map, now thought of as an R-module homomorphism. A short exact sequence in S-**Mod** becomes, via this functor, a short exact sequence in R-**Mod**, and so Γ is exact. Thus there is a monoid homomorphism $\gamma\colon M(S\text{-}\mathbf{Mod}) \to M(R\text{-}\mathbf{Mod})$ such that $\gamma([{}_SA]) = [{}_RA]$ for all ${}_SA \in S\text{-}\mathbf{Mod}$.

One special case of this is the functor arising from the quotient map $R \to R/I$ where I is a two-sided ideal of R. This functor induces a monoid homomorphism $\gamma_I\colon M(R/I\text{-}\mathbf{Mod}) \to M(R\text{-}\mathbf{Mod})$ with important properties:

Lemma 3.5. [4, 3.9][5, 4.6] *Let R be a ring, $I \subseteq R$ a two-sided ideal and $\gamma_I\colon M(R/I\text{-}\mathbf{Mod}) \to M(R\text{-}\mathbf{Mod})$ as above.*

(1) *γ_I is injective. In particular, $M(R/I\text{-}\mathbf{Mod}) \cong \gamma_I(M(R/I\text{-}\mathbf{Mod}))$ and $M(R/I\text{-}\mathbf{Noeth}) \cong \gamma_I(M(R/I\text{-}\mathbf{Noeth}))$.*
(2) *$\gamma_I(M(R/I\text{-}\mathbf{Mod}))$ is an order ideal of $M(R\text{-}\mathbf{Mod})$.*
(3) *$\gamma_I(M(R/I\text{-}\mathbf{Noeth}))$ is an order ideal of $M(R\text{-}\mathbf{Noeth})$.*
(4) *If I is nilpotent, then γ_I is surjective, $M(R/I\text{-}\mathbf{Mod}) \cong M(R\text{-}\mathbf{Mod})$ and $M(R/I\text{-}\mathbf{Noeth}) \cong M(R\text{-}\mathbf{Noeth})$.*

The Serre subcategories of R-**Mod** which are obtained from the order ideals $\gamma_I(M(R/I\text{-}\mathbf{Mod}))$ and $\gamma_I(M(R/I\text{-}\mathbf{Noeth}))$ using 3.4(1) are

$$\{\, A \in R\text{-}\mathbf{Mod} \mid \exists n \in \mathbb{N} \text{ such that } I^nA = 0 \,\}$$

and

$$\{\, A \in R\text{-}\mathbf{Noeth} \mid \exists n \in \mathbb{N} \text{ such that } I^nA = 0 \,\},$$

respectively. In view of this lemma, we will make the convention that, for any two-sided ideal I of R, $M(R/I\text{-}\mathbf{Mod})$ and $M(R/I\text{-}\mathbf{Noeth})$ are order ideals of $M(R\text{-}\mathbf{Mod})$.

For the remainder of this section we specialize to commutative rings. Our immediate goal is to show that if P is a prime ideal of a commutative ring R, then $[R/P]$ is a free prime element of the monoid $M(R\text{-}\mathbf{Mod})$.

Lemma 3.6. *Let P be a prime ideal of a commutative ring R and $\mathbb{A}$ a partition of R/P.*

(1) $\mathbb{A}$ *has a refinement containing* R/P.
(2) $\mathbb{A}$ *contains at most one module isomorphic to* R/P.

Proof. We note first that one of the nonzero modules in $\mathbb{A}$, A_0 say, must be isomorphic to I/P for some ideal I containing P.

(1) If $x \in I \setminus P$, then $(Rx+P)/P \cong R/P$ and so A_0 has the partition $(R/P, I/(Rx+P))$. Then A_0 can be replaced by $(R/P, I/(Rx+P))$ in $\mathbb{A}$ to yield a refinement of $\mathbb{A}$ containing R/P.
(2) All modules in $\mathbb{A}$ except A_0 are isomorphic to subfactor modules of R/I and hence their annihilators contain I. Consequently none of these modules can be isomorphic to R/P. □

For $n \in \mathbb{N}$, we will write $\mathbb{P}_n$ for the partition $(R/P, R/P, \dots, R/P)$ with n copies of R/P.

Theorem 3.7. *Let P be a prime ideal of a commutative ring R. Then $[R/P]$ is a free prime element of $M(R$-**Mod**$)$. Moreover, for $A \in R$-**Mod** and $\mathbb{P}_n$ as above, we have*

$$\phi_{[R/P]}([A]) = \sup\{\, n \in \mathbb{Z}^+ \mid \mathbb{P}_n \textit{ is a subpartition of } A\,\}.$$

Proof. For $A \in R$-**Mod**, define

$$\phi(A) = \sup\{\, n \in \mathbb{Z}^+ \mid \mathbb{P}_n \text{ is a subpartition of } A\,\}.$$

Suppose we have a short exact sequence $0 \to A \to B \to C \to 0$ in R-**Mod**. If $\mathbb{P}_m$ is a subpartition of a partition $\mathbb{A}$ of A, and $\mathbb{P}_n$ is a subpartition of a partition $\mathbb{C}$ of C, then by concatenating $\mathbb{A}$ and $\mathbb{C}$ to form a partition of B, we see that $\mathbb{P}_{m+n}$ is a subpartition of B. Thus $m+n \le \phi(B)$ and, taking the supremum over all such m and n, we get $\phi(A)+\phi(C) \le \phi(B)$.

Conversely, suppose $\mathbb{P}_k$ is a subpartition of a partition $\mathbb{B}$ of B. Let $\mathbb{B}'$ be a common refinement of (A, C) and $\mathbb{B}$. Then $\mathbb{B}'$ contains k (nonoverlapping) partitions of R/P. Using 3.6(1) we can make further refinements to get a partition $\mathbb{B}''$ which is a refinement of (A, C) and has $\mathbb{P}_k$ as a subpartition. From this it follows that there are $m, n \in \mathbb{Z}^+$ such that $k = m+n$, $\mathbb{P}_m$ is a subpartition of A, and $\mathbb{P}_n$ is a subpartition of C. Thus $k \le \phi(A)+\phi(C)$ and, taking the supremum over all such k, we get $\phi(B) \le \phi(A)+\phi(C)$.

Since ϕ respects short exact sequences, there is an induced monoid homomorphism from $M(R$-**Mod**$)$ to $\mathbb{Z}^\infty$, which we will also call ϕ, such that $\phi([A]) = \phi(A)$ for all $A \in R$-**Mod**. Notice that from 3.6(2) we get $\phi([R/P]) = \phi(R/P) = 1$.

Showing that $[R/P]$ is a free prime is now easy: If $[R/P] \le [A_1]+[A_2]$, then, applying ϕ, we get $1 = \phi(R/P) \le \phi(A_1)+\phi(A_2)$. Thus for some $i \in \{1,2\}$ we have $1 \le \phi(A_i)$. From the definition of ϕ and 3.3(5) it follows that $[R/P] \leq [A_i]$. Thus $[R/P]$ is prime. Further, if $m[R/P] \leq n[R/P]$ for some $m, n \in \mathbb{N}$, then applying ϕ we get $m \leq n$. Thus $[R/P]$ is free.

For $A \in R$-**Mod**, to prove that $\phi_{[R/P]}([A]) = \phi(A)$, it suffices to show that, for any $n \in \mathbb{Z}^+$, $n[R/P] \leq [A]$ if and only if $\mathbb{P}_n$ is a subpartition of A. But if $n[R/P] \leq [A]$ then applying ϕ we get $n \leq \phi(A)$ which implies $\mathbb{P}_n$ is a subpartition of A. The converse implication is immediate from 3.3(5). $\square$

We now specialize further to commutative Noetherian rings.

Lemma 3.8. *Let R be a commutative Noetherian ring.*

(1) $M(R\text{-}\mathbf{Noeth}) = \{\propto [R]\}$
(2) *If $A \in R$-**Noeth** is nonzero, there are prime ideals $P_1, P_2, \ldots, P_n$ of R such that $[A] = [R/P_1] + [R/P_2] + \ldots + [R/P_n]$.*
(3) *If $U \in R$-**Noeth** is nonzero such that $[U]$ is prime, then $[U] \equiv [R/P]$ where P is a prime ideal of R.*

Proof.

(1) Since R is Noetherian, we have $[R] \in M(R\text{-}\mathbf{Noeth})$, and so the order ideal generated by $[R]$, namely $\{\propto [R]\}$, is contained in $M(R\text{-}\mathbf{Noeth})$. Conversely if $A \in R$-**Noeth**, then A is finitely generated and so, from 3.3(6), $[A] \propto [R]$.
(2) The module A has a submodule series $0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = A$ such that for $i = 1, 2, \ldots, n$, $A_i/A_{i-1} \cong R/P_i$ for some prime ideal P_i [16, Theorem 6.4]. Thus $(R/P_1, R/P_2, \ldots, R/P_n)$ is a partition of A and the claim follows from 3.3(5).
(3) From 2, $[U] = [R/P_1] + [R/P_2] + \ldots + [R/P_n]$ for some prime ideals $P_1, P_2, \ldots, P_n$. Since $[U]$ is prime there must be some index i such that $[U] \equiv [R/P_i]$. $\square$

Theorem 3.9. *If R is a commutative Noetherian ring, then $M(R$-**Noeth**$)$ is primely generated, strongly separative and has the properties P1-P7 of 2.13.*

Proof. The element $0 = [0] \in M(R\text{-}\mathbf{Noeth})$ is prime, and for any module $A \in R$-**Noeth** which is nonzero, we have from 3.8(2) and 3.7 that $[A]$ is a sum of primes. Therefore $M(R\text{-}\mathbf{Noeth})$ is primely generated by the set $\{\, [R/P] \mid P \in \operatorname{Spec} R \,\} \cup \{0\}$.

We have from 3.7 that for any p in this set of generators, either p is free or $p = 0$. Thus all the remaining claims follow from Theorem 2.13. $\square$

In [4, 5.1] it is shown that for *any* ring R, $M(R\text{-}\mathbf{Noeth})$ is strongly separative. It is not known which, if any, of the other properties in 2.13 occur in this generality.

4. The Radann map

The radann map takes R-modules to radical ideals of the ring R and respects exact sequences. With it we will be able to link the prime elements of the monoid $M(R\text{-}\mathbf{Mod})$ with the prime ideals of the ring.

Definition 4.1. *Let R be a commutative ring,* $\operatorname{Spec} R$ *the set of all prime ideals of R, and* $\operatorname{Rad} R$ *the set of all radical ideals of R, that is, all intersections of sets of prime ideals. R is a radical ideal by this definition since it is the intersection of the empty set of prime ideals.*

For an ideal $I \subseteq R$ we define the **radical** *of I by*

$$\operatorname{rad} I = \bigcap \{ P \in \operatorname{Spec} R \mid I \subseteq P \} \in \operatorname{Rad} R.$$

An equivalent definition is $\operatorname{rad} I = \{ r \in R \mid \exists n \in \mathbb{N}$ *such that* $r^n \in I \}$.

For a module $A \in R$-$\mathbf{Mod}$ we define

$$\operatorname{radann} A = \operatorname{rad}(\operatorname{ann} A)$$

where $\operatorname{ann} A = \{ r \in R \mid rA = 0 \}$ *is the annihilator of A.*

If I is an ideal of R, then we have immediately that $I \subseteq \operatorname{rad} I$ and $(I \in \operatorname{Rad} R \iff \operatorname{rad} I = I)$. Further if $I_1 \subseteq I_2$ are ideals of R, then $\operatorname{rad} I_1 \subseteq \operatorname{rad} I_2$.

Lemma 4.2. *Let I, I_1, I_2 be ideals of R, and $A, B, C \in R$-$\mathbf{Mod}$.*

(1) $\operatorname{rad} I = R \iff I = R$
(2) $A = 0 \iff [A] = 0 \iff \operatorname{ann} A = R \iff \operatorname{radann} A = R$
(3) $\operatorname{rad} I_1 \cap \operatorname{rad} I_2 = \operatorname{rad}(I_1 \cap I_2) = \operatorname{rad}(I_1 I_2)$
(4) *If $0 \to A \to B \to C \to 0$ is exact, then*

$$(\operatorname{ann} A)(\operatorname{ann} C) \subseteq \operatorname{ann} B \subseteq \operatorname{ann} A \cap \operatorname{ann} C$$
$$\operatorname{radann} B = \operatorname{radann} A \cap \operatorname{radann} C.$$

Proof.

(1) If $I \neq R$, then there is a maximal, and hence prime, ideal P such that $I \subseteq P \neq R$. Thus $\operatorname{rad} I \subseteq P$ and $\operatorname{rad} I \neq R$.
(2) From (1) and 3.3(1).
(3) [2, II.2.6 Cor. 2]
(4) The first claim is easy. Then the second claim follows from (3). □

The last claim of this lemma suggests that we should consider $\operatorname{Rad} R$ to be a monoid with the operation $\cap$ and identity R, so that radann respects exact sequences as a map to $(\operatorname{Rad} R, \cap, R)$. From the universal property of $M(R\text{-}\mathbf{Mod})$, we have a monoid homomorphism, which we will also call radann, from $M(R\text{-}\mathbf{Mod})$ to $\operatorname{Rad} R$ such that $\operatorname{radann}[A] = \operatorname{radann} A$ for all $A \in R$-**Mod**. The homomorphism $\operatorname{radann}\colon M(R\text{-}\mathbf{Mod}) \to \operatorname{Rad} R$ is surjective since for any $S \in \operatorname{Rad} R$ we have

$$\operatorname{radann}([R/S]) = \operatorname{radann}(R/S) = \operatorname{rad} S = S.$$

It is easy to check that the preorder $\leq$ on the monoid $\operatorname{Rad} R$ is reverse inclusion:

$$S_1 \leq S_2 \iff S_1 \supseteq S_2$$

for all $S_1, S_2 \in \operatorname{Rad} R$.

Lemma 4.3. *If I is an ideal of R and $S \in \operatorname{Rad} R$, then*

$$[R/S] \leq [R/I] \iff S \supseteq I.$$

In particular, if P_1, P_2 are prime ideals of R, then

$$[R/P_1] \leq [R/P_2] \iff P_1 \supseteq P_2.$$

Proof. Since radann is a homomorphism, from $[R/S] \leq [R/I]$, we get

$$S = \operatorname{radann}[R/S] \supseteq \operatorname{radann}[R/I] = \operatorname{rad} I \supseteq I.$$

The converse is immediate since $S \supseteq I$ implies that R/S is a quotient module of R/I. □

The monoid $(\operatorname{Rad} R, \cap, R)$ has the property that for all $S \in \operatorname{Rad} R$, $S \cap S = S$, that is, every element is idempotent. Such monoids are called 0-semilattices:

A **0-semilattice** [6, Section 3], [12, 1.3.2] is a partially ordered set $(\mathcal{L}, \leq)$ with a minimum element 0 such that $a \vee b$ exists for all $a, b \in \mathcal{L}$. In this circumstance $(\mathcal{L}, \vee, 0)$ is a monoid in which the preorder $\leq$ defined as in 2.1 coincides with the given partial order on $\mathcal{L}$. Conversely , if M is a monoid such that $2a = a$ for all $a \in M$, then $(M, \leq)$ is a 0-semilattice with minimum element 0 in which $+$ and $\vee$ coincide.

To prove that $(\operatorname{Rad} R, \cap, R)$ has refinement, it will be convenient to define $\mathcal{K}(I) = \{\, P \in \operatorname{Spec} R \mid I \subseteq P \,\}$ for any ideal I of a ring R. Then for ideals I and J we have $(I \subseteq J \implies \mathcal{K}(I) \supseteq \mathcal{K}(J))$, $\mathcal{K}(I \cap J) = \mathcal{K}(I) \cup \mathcal{K}(J)$, and $(I \in \operatorname{Rad} R \iff I = \bigcap \mathcal{K}(I))$.

Lemma 4.4. *For any commutative ring R, $(\operatorname{Rad} R, \cap, R)$ is a refinement monoid. Moreover, an ideal $S \in \operatorname{Rad} R$ is prime element of the monoid if and only if $S = R$ or S is a prime ideal.*

Proof. Suppose we have $A_1, A_2, B_1, B_2 \in \operatorname{Rad} R$ with $A_1 \cap A_2 = B_1 \cap B_2$. Set $C_{ij} = \bigcap(\mathcal{K}(A_i) \cap \mathcal{K}(B_j)) \in \operatorname{Rad} R$ for $i, j = 1, 2$. Thus, by construction,

$\mathcal{K}(C_{ij}) = \mathcal{K}(A_i) \cap \mathcal{K}(B_j)$. To prove that $A_1 = C_{11} \cap C_{12}$, it suffices to show $\mathcal{K}(A_1) = \mathcal{K}(C_{11} \cap C_{12})$:

$$\begin{aligned}\mathcal{K}(C_{11} \cap C_{12}) &= \mathcal{K}(C_{11}) \cup \mathcal{K}(C_{12}) \\ &= (\mathcal{K}(A_1) \cap \mathcal{K}(B_1)) \cup (\mathcal{K}(A_1) \cap \mathcal{K}(B_2)) \\ &= \mathcal{K}(A_1) \cap (\mathcal{K}(B_1) \cup \mathcal{K}(B_2)) \\ &= \mathcal{K}(A_1) \cap (\mathcal{K}(B_1 \cap B_2)).\end{aligned}$$

But $B_1 \cap B_2 \subseteq A_1$, hence $\mathcal{K}(B_1 \cap B_2) \supseteq \mathcal{K}(A_1)$ and $\mathcal{K}(C_{11} \cap C_{12}) = \mathcal{K}(A_1)$.

Similarly, $A_2 = C_{21} \cap C_{22}$, $B_1 = C_{11} \cap C_{21}$ and $B_2 = C_{12} \cap C_{22}$ and so $\operatorname{Rad} R$ has refinement.

Suppose now S is a prime element of $\operatorname{Rad} R$ and $I_1, I_2 \subseteq R$ are ideals such that $I_1 I_2 \subseteq S$. Using 4.2(3), we get $\operatorname{rad} I_1 \cap \operatorname{rad} I_2 = \operatorname{rad}(I_1 I_2) \subseteq \operatorname{rad} S = S$, or, as elements of the monoid, $\operatorname{rad} I_1 \cap \operatorname{rad} I_2 \geq S$. Since S is a prime element of the monoid, either $\operatorname{rad} I_1 \geq S$ or $\operatorname{rad} I_2 \geq S$. Hence, either $I_1 \subseteq \operatorname{rad} I_1 \subseteq S$ or $I_2 \subseteq \operatorname{rad} I_2 \subseteq S$. This makes S either a prime ideal of R or R itself.

Conversely, suppose that S is a prime ideal and $S_1, S_2 \in \operatorname{Rad} R$ such that $S_1 \cap S_2 \geq S$, that is, $S_1 \cap S_2 \subseteq S$. Then $S_1 S_2 \subseteq S$ so either $S_1 \subseteq S$ or $S_2 \subseteq S$, that is, either $S_1 \geq S$ or $S_2 \geq S$. Thus S is a prime element of $\operatorname{Rad} R$

Note also that R is the identity of the monoid, so it is a prime element of $\operatorname{Rad} R$. □

It is interesting to notice that $(\operatorname{Rad} R, \leq)$ is not just a semilattice, but also a lattice with

$$S_1 \vee S_2 = S_1 \cap S_2 \qquad S_1 \wedge S_2 = \operatorname{rad}(S_1 + S_2)$$

for all $S_1, S_2 \in \operatorname{Rad} R$. The fact that $(\operatorname{Rad} R, \cap, R)$ has refinement implies that $(\operatorname{Rad} R, \vee, \wedge)$ is a distributive lattice. See [6, 3.6] and [11, page 99] for the details.

If R is Noetherian, then any $S \in \operatorname{Rad} R$ is actually an intersection of a finite number of prime ideals, and so $\operatorname{Rad} R$ is a primely generated refinement monoid. We notice further that in this case, if $[U] \in M(R\text{-}\mathbf{Noeth})$ is prime, then either $[U] = 0$ and $\operatorname{radann}[U] = R$, or, by 3.8(3), $[U] \equiv [R/P]$ for some prime ideal P and $\operatorname{radann}[U] = P$. Thus radann maps prime elements of $M(R\text{-}\mathbf{Noeth})$ to prime elements of $\operatorname{Rad} R$.

Theorem 4.5. *Let R be a commutative Noetherian ring. Then for all $A, B \in R$-**Noeth***

(1) $[A] \asymp [R/\operatorname{ann} A] \asymp [R/\operatorname{radann} A]$
(2) $[A] \propto [B] \iff \operatorname{radann} A \supseteq \operatorname{radann} B$

(3) $[A] \asymp [B] \iff \operatorname{radann} A = \operatorname{radann} B$

Proof.

(1) If $A = 0$, then $[A] = [R/\operatorname{ann} A] = [R/\operatorname{radann} A] = 0$ and the claim is true. Otherwise, from 3.8(2), there are prime ideals $P_1, P_2, \dots, P_n$ of R such that $[A] = [R/P_1] + [R/P_2] + \dots + [R/P_n]$. For $i = 1, 2, \dots, n$, we have $[R/P_i] \le [A]$, so

$$P_i = \operatorname{radann}[R/P_i] \supseteq \operatorname{radann}[A] = \operatorname{radann} A$$

and hence $[R/P_i] \le [R/\operatorname{radann} A]$. Thus $[A] \le n[R/\operatorname{radann} A]$ and, in particular, $[A] \propto [R/\operatorname{radann} A]$.

Since $\operatorname{ann} A \subseteq \operatorname{radann} A$, we have $[R/\operatorname{radann} A] \le [R/\operatorname{ann} A]$ and hence also $[R/\operatorname{radann} A] \propto [R/\operatorname{ann} A]$.

Let $\{a_1, a_2, \dots, a_n\}$ be a set of generators for A, and $\phi\colon R \to A^n$ the homomorphism defined by $\phi(r) = (ra_1, ra_2, \dots, ra_n)$. Then $\ker\phi = \operatorname{ann} A$, and so $R/\operatorname{ann} A$ is isomorphic to a submodule of A^n and $[R/\operatorname{ann} A] \le n[A]$. This implies $[R/\operatorname{ann} A] \propto [A]$.

(2) If $\operatorname{radann} A \supseteq \operatorname{radann} B$, then $[R/\operatorname{radann} A] \le [R/\operatorname{radann} B]$ and hence, using (1), $[A] \propto [B]$.

The converse is easy since radann is a homomorphism and $\operatorname{Rad} R$ a semilattice.

(3) Follows directly from (2). □

Since the radann map is surjective, item 3 of this theorem implies that, for a commutative Noetherian ring R,

$$\operatorname{Rad} R \cong M(R\text{-}\mathbf{Noeth})/\asymp .$$

It is easy to show that for any monoid M, the quotient monoid $M/\asymp$ has the following universal property with respect to 0-semilattices: If $\phi\colon M \to \mathcal{L}$ is a monoid homomorphism with $\mathcal{L}$ a 0-semilattice, then ϕ factors uniquely through the quotient map from M to $M/\asymp$. If $\operatorname{Rad} R \cong M(R\text{-}\mathbf{Noeth})/\asymp$, then the monoid $\operatorname{Rad} R$ has this same universal property:

Corollary 4.6. *If R is a commutative Noetherian ring, $\mathcal{L}$ a 0-semilattice, and $\Phi\colon R\text{-}\mathbf{Noeth} \to \mathcal{L}$ a map which respects short exact sequences, then there is a unique monoid homomorphism $\phi\colon \operatorname{Rad} R \to \mathcal{L}$ such that, for all $A \in R\text{-}\mathbf{Noeth}$, $\Phi(A) = \phi(\operatorname{radann} A)$.*

To be explicit, the map $\Phi\colon R\text{-}\mathbf{Noeth} \to \mathcal{L}$ respects short exact sequences if $\Phi(0) = 0 \in \mathcal{L}$, and $\Phi(B) = \Phi(A) \vee \Phi(C)$ whenever $0 \to A \to B \to C \to 0$ is an exact sequence in R-**Noeth**.

5. The structure of $\{\asymp [A]\}$

In the previous section we found that, when R is a commutative Noetherian ring , then $M(R\text{-}\mathbf{Noeth})$ modulo the congruence $\asymp$ is isomorphic to $\operatorname{Rad} R$. The obvious next question is: What is the structure of a typical congruence class $\{\asymp a\}$ where $a = [A]$ for some nonzero $A \in R\text{-}\mathbf{Noeth}$?

Since $M(R\text{-}\mathbf{Noeth})$ is separative, we know from 2.3(S3) that $\{\asymp a\}$ is cancellative. Since $M(R\text{-}\mathbf{Noeth})$ is strongly separative, we have from 2.5(2) that a is free, and so 2.12 applies and we have the more precise information that $\{\asymp a\} \cong \mathbb{N}^n \times G_a$ where $n \in \mathbb{N}$ is the number of elements in an incomparable set of prime elements $\{p_1, p_2, \ldots, p_n\} \subseteq M(R\text{-}\mathbf{Noeth})$ such that

$$a \equiv \phi_{p_1}(a)p_1 + \phi_{p_2}(a)p_2 + \ldots + \phi_{p_n}(a)p_n.$$

We will show in this section that n has a more familiar module theoretic meaning, namely, it is the number of prime ideals minimal among the associated primes of A, or equivalently, n is the number of prime ideals minimal in the support of A.

Given a module $A \in R\text{-}\mathbf{Noeth}$, a prime ideal P such that $P = \operatorname{ann} a$ for some $a \in A$ is called an **associated prime** of A. The set of associated primes of A is written $\operatorname{Ass} A$. The set of prime ideals P such that the localization A_P is nonzero, is called the **support** of A and is written $\operatorname{Supp} A$.

Lemma 5.1. *Let R be a commutative Noetherian ring, P a prime ideal and $A \in R\text{-}\mathbf{Noeth}$. Then the following are equivalent:*

(1) $P \in \operatorname{Supp} A$
(2) $\operatorname{ann} A \subseteq P$
(3) *A has a subfactor isomorphic to R/P*
(4) $1 \leq \phi_{[R/P]}([A])$
(5) $[R/P] \leq [A]$
(6) $[R/P] \propto [A]$

Proof. The equivalence of (1) and (2) is proved in [16, page 26]. The equivalence of (3), (4) and (5) comes directly from 3.7. Since $[R/P]$ is prime, (5) and (6) are easily seen to be equivalent. If A has a subfactor isomorphic to R/P, then $\operatorname{ann} A \subseteq \operatorname{ann} R/P = P$, thus (3) implies (2). To complete the proof we show that (2) implies (6):

Since P is a prime ideal, $\operatorname{ann} A \subseteq P$ implies that $\operatorname{radann} A \subseteq P$. We also have $P = \operatorname{radann} R/P$ and so, from 4.5(2), $[R/P] \propto [A]$. $\square$

In the next theorem we will assume that $\operatorname{Spec} R$ is ordered by inclusion, rather than reverse inclusion which would be appropriate if one was considering $\operatorname{Spec} R$ as a subset of the monoid $\operatorname{Rad} R$.

Theorem 5.2. *Let R be a commutative Noetherian ring. Then for any nonzero module $A \in R$-**Noeth** the following sets coincide:*

(1) *The set of minimal elements of* $\operatorname{Ass} A$
(2) *The set of minimal elements of* $\operatorname{Supp} A$
(3) *The set of minimal elements of* $\{P_1, P_2, \dots, P_m\} \subseteq \operatorname{Spec} R$ *whenever* $[A] = [R/P_1] + [R/P_2] + \ldots + [R/P_m]$.
(4) *The unique incomparable set* $\{P_1, P_2, \dots, P_n\} \subseteq \operatorname{Spec} R$ *such that* $[A] \equiv n_1[R/P_1] + n_2[R/P_2] + \ldots + n_n[R/P_n]$ *with* $n_1, n_2, \dots, n_n \in \mathbb{N}$.
(5) *The unique incomparable set* $\{P_1, P_2, \dots, P_k\} \subseteq \operatorname{Spec} R$ *such that* $\operatorname{radann} A = \cap_i P_i$.

Proof. In [16, 6.5(iii)] it is proved that the first two sets are equal.

Suppose there exist finite subsets $X, Y, Z \subseteq \operatorname{Spec} R$ such that

- $[A] = \sum_{P \in X} [R/P]$
- $[A] \equiv \sum_{P \in Y} n_P [R/P]$ with $n_P \in \mathbb{N}$ for all $P \in Y$
- $\operatorname{radann} A = \cap_{P \in Z} P$.

We will show that every element of $\operatorname{Supp} A$ contains an element of X, every element of X contains an element of Y, every element of Y contains an element of Z, and every element of Z contains an element of $\operatorname{Supp} A$. This has as a consequence that $\operatorname{Supp} A$, X, Y and Z have the same set of minimal elements, and shows uniqueness in case one of these sets is incomparable.

Suppose $P_0 \in \operatorname{Supp} A$. Then, from 5.1, we have $[R/P_0] \leq [A]$, and, since $[A] = \sum_{P \in X} [R/P]$ and $[R/P_0]$ is prime, there is some $P_1 \in X$ such that $[R/P_0] \leq [R/P_1]$. From 4.3, this implies $P_1 \subseteq P_0$.

Suppose $P_1 \in X$. Then $[R/P_1] \leq [A]$ with $[A] \equiv \sum_{P \in Y} n_P [R/P]$, so just as in the previous paragraph, there is some $P_2 \in Y$ such that $P_2 \subseteq P_1$.

Suppose $P_2 \in Y$. Then $[R/P_2] \leq [A]$ and so

$$\prod_{P \in Z} P \subseteq \bigcap_{P \in Z} P = \operatorname{radann} A \subseteq \operatorname{radann} R/P_2 = P_2.$$

Since P_2 is prime, there is some $P_3 \in Z$ such that $P_3 \subseteq P_2$.

Suppose $P_3 \in Z$. Then we have $\operatorname{radann} A \subseteq P_3 = \operatorname{radann} R/P_3$, and, from 4.5(2), $[R/P_3] \propto [A]$. Since $[R/P_3]$ is prime, this implies $[R/P_3] \leq [A]$ and so, from 5.1, $P_3 \in \operatorname{Supp} A$.

Now we consider the question of whether the sets described in (4) and (5) exist. From 3.8(2), there are prime ideals $P_1, P_2, \dots, P_n$ of R such that $[A] = [R/P_1] + [R/P_2] + \ldots + [R/P_n]$. Collecting equal terms together we can certainly write $[A] \equiv n_1[R/P_1] + n_2[R/P_2] + \ldots + n_n[R/P_n]$ where now $P_1, P_2, \ldots, P_n$ are distinct primes and $n_1, n_2, \ldots, n_n \in \mathbb{N}$. We will show that we can drop any terms in this sum which correspond to primes which are not minimal in the set $\{P_1, P_2, \ldots, P_n\}$.

If we had, for example, $P_1 \subseteq P_2$ (and $P_1 \neq P_2$), then $[R/P_2] \leq [R/P_1]$ and so $[R/P_2] + [B] = [R/P_1]$ for some $B \in R$-**Noeth**. Since $[R/P_1]$ is prime, but, by 4.3, $[R/P_1] \not\leq [R/P_2]$, we must have $[R/P_1] \leq [B]$. It follows easily that $[R/P_1] + [R/P_2] \equiv [R/P_1]$, so $n_1[R/P_1] + n_2[R/P_2] \equiv n_1[R/P_1]$ and we can drop the term $n_2[R/P_2]$ from the expression for A.

Removing all such terms, and relabeling we have an incomparable set of primes $P_1, P_2, \ldots, P_n$ such that $[A] \equiv n_1[R/P_1] + n_2[R/P_2] + \ldots + n_n[R/P_n]$ for some $n_1, n_2, \ldots n_n \in \mathbb{N}$. Applying radann to this equation one gets $\operatorname{radann} A = \operatorname{radann}[A] = \cap_i \operatorname{radann}(n_i[R/P_i]) = \cap_i P_i$. $\square$

It follows easily from 4.3 that a set of prime ideals $\{P_1, P_2, \ldots, P_n\}$ is incomparable if and only if the set $\{[R/P_1], [R/P_1], \ldots, [R/P_n]\}$ is incomparable. Comparison of (4) of this theorem with Lemma 2.12 yields the following:

Corollary 5.3. *Let A be a nonzero Noetherian module over a commutative Noetherian ring R. Then*

$$[A] \equiv \phi_{[R/P_1]}([A])[R/P_1] + \phi_{[R/P_2]}([A])[R/P_2] + \ldots + \phi_{[R/P_n]}([A])[R/P_n]$$

where $\{P_1, P_2, \ldots, P_n\}$ is the incomparable set of prime ideals provided by the theorem. Moreover, $\{\asymp [A]\} \cong \mathbb{N}^n \times G_{[A]}$ (as semigroups).

Theorem 5.4. *Let A and B be Noetherian modules over a commutative Noetherian ring R. Then $[A] = [B]$ if and only if the following three conditions hold:*

(1) $\operatorname{radann} A = \operatorname{radann} B$. *Equivalently, the incomparable set of prime ideals $\{P_1, P_2, \ldots, P_n\}$ obtained from A using 5.2, coincides with the incomparable set of prime ideals obtained from B. This condition implies that $[A] \asymp [B]$ and $G_{[A]} = G_{[B]}$.*
(2) $\phi_{[R/P_i]}([A]) = \phi_{[R/P_i]}([B])$ *for $i = 1, 2, \ldots, n$. Conditions (1) and (2) imply $[A] \equiv [B]$.*
(3) *A and B represent the same element in $G_{[A]} = G_{[B]}$. More precisely, given an arbitrary element $u \equiv [A] \equiv [B]$, we have $G_u = G_{[A]} = G_{[B]}$, and via 2.7(3) we can think of $[A]$ and $[B]$ as elements of the group $(\{\equiv u\}, \square_u, u) \cong G_u$. We require then that $[A] = [B]$ in this group.*

Proof. Of course since radann and $\phi_{[R/P_i]}$ are homomorphisms the necessity of these conditions is clear.

From 5.2(5) we have the equivalence of the conditions in (1). The remaining claims in (1) are from 4.5(3), 3.9 and 2.7(2).

If A and B satisfy conditions (1) and (2), then, from 5.3 we have $[A] \equiv [B]$.

We now can think of $[A]$ and $[B]$ as elements of the group $(\{\equiv u\}, \Box_u, u)$, so the last requirement, that $[A] = [B]$ in this group, means $[A] = [B]$ in $M(R\text{-}\mathbf{Noeth})$. $\Box$

6. The structure of $G_{[A]}$

For a commutative Noetherian ring R, Theorem 5.4 reduces the problem of understanding the structure of $M(R\text{-}\mathbf{Noeth})$ to the understanding of the Abelian groups $G_{[A]}$ for $A \in R\text{-}\mathbf{Noeth}$. Unfortunately, there is not much that can be said about these groups in general — on the positive side, this means that these groups contain potentially important information about modules. If R is a Dedekind domain we will show that $G_{[R]}$ is isomorphic to the ideal class group of the ring. This suggests that, in general, calculating $G_{[A]}$ will be at least as difficult as calculating the ideal class groups of a Dedekind domain.

In our discussion we will often take advantage of the alternative description of $G_{[A]}$ as the set $\{\equiv [A]\}$ with operation $\Box_{[A]}$ and identity $[A]$ as described in 2.7(3).

For example, let I be an ideal. Then from 3.5, $M(R/I\text{-}\mathbf{Noeth})$ embeds as an order ideal in $M(R\text{-}\mathbf{Noeth})$. Considering R/I as a ring, R-module or R/I-module as appropriate, it is easy to check that the sets $\{\equiv [R/I]\}$ and operation $\Box_{[R/I]}$ have the same meaning in both $M(R\text{-}\mathbf{Noeth})$ and $M(R/I\text{-}\mathbf{Noeth})$. Thus $G_{[R/I]}$ defined relative to $M(R\text{-}\mathbf{Noeth})$ is isomorphic to $G_{[R/I]}$ defined relative to $M(R/I\text{-}\mathbf{Noeth})$.

From 2.6 there is an Abelian group $G_{[A]}$ for every module $A \in R\text{-}\mathbf{Noeth}$. There are not quite so many groups as one might think from this statement: From 4.5 we have that $[A] \asymp [R/S]$ where $S = \operatorname{radann} A$, and hence, from 2.7(2), $G_{[A]} = G_{[R/S]}$. Thus we have at most one group for each radical ideal of R.

Further, if $A \neq 0$, then from 3.8(2), $[A] = [R/P_1] + [R/P_2] + \ldots + [R/P_n]$ for some prime ideals $P_1, P_2, \ldots, P_n$. Since 2.14 applies to $M(R\text{-}\mathbf{Noeth})$, we have that $G_{[A]}$ is a homomorphic image of the direct sum of the groups $G_{[R/P_i]}$ for $i = 1, 2, \ldots, n$. Thus we will focus on groups of the form $G_{[R/P]}$ where P is a prime ideal of R.

Suppose first that P is a maximal ideal, so that R/P is a simple module. Then it is trivial that $[B] \equiv [R/P] \iff [B] = [R/P] \iff B \cong R/P$ for any module B, and so $\{\equiv [R/P]\} = \{[R/P]\}$ and $G_{[R/P]}$ is trivial.

If $A \in R$-**Noeth** is a finite length module, then there are simple modules $A_1, A_2, \ldots, A_n$ such that $[A] = [A_1] + [A_2] + \ldots + [A_n]$. From 2.14, the group $G_{[A]}$ is a homomorphic image of the direct sum of the groups $G_{[A_i]}$ for $i = 1, 2, \ldots, n$. Since each of these groups is trivial, so is $G_{[A]}$.

That $G_{[A]}$ is trivial when A has finite length is true, in fact, over any ring. This can be seen as an easy consequence of the structure of $M(R$-**Len**$)$ discussed in [5, 4.8]. Here R-**Len** is the Serre subcategory of finite length R-modules.

The next step would be to discuss $G_{[R/P]}$ when R/P has dimension 1. From the above discussion $G_{[R/P]}$ defined relative to $M(R$-**Noeth**$)$ is isomorphic to $G_{[R/P]}$ defined relative to $M(R/P$-**Noeth**$)$, so without loss of generality, we can assume that R is a 1-dimensional Noetherian domain and we are interested in $G_{[R]}$.

We notice first that $G_{[R]}$ is the only (potentially) non-trivial group built into $M(R$-**Noeth**$)$. Indeed, if $A \in R$-**Noeth** and $1 \leq \phi_{[R]}([A])$, then from 5.1 and 3.8(1) we get $[A] \asymp [R]$, and then by 3.9 and 2.7(2), $G_{[A]} = G_{[R]}$. On the other hand, if $0 = \phi_{[R]}([A])$, then by 5.1, $\operatorname{ann} A \neq 0$, which implies A has finite length and so $G_{[A]}$ is trivial.

For dimension 1 rings we can quote the following result which provides generators and relations for $G_{[R]}$:

Theorem 6.1. [5, 1.1,6.6(6)] *Let R be a commutative Noetherian domain with dimension* 1*, and $\mathbb{S}$ a set of representatives of the isomorphism classes of simple R-modules. Then $G_{[R]}$ is isomorphic to the Abelian group with one generator $\langle S \rangle$ for each $S \in \mathbb{S}$, and relations $\langle S_1 \rangle + \langle S_2 \rangle + \ldots + \langle S_k \rangle = 0$ whenever $S_1, S_2, \ldots, S_k \in \mathbb{S}$ are isomorphic to the composition series factors of $R/(x)$ for some irreducible $x \in R$.*

Remark: From [5], we could have simply required that $x \neq 0$ in this theorem — choosing x to be irreducible just removes redundant relations.

The main goal of this section is to prove that, in the special case that R is a Dedekind domain, $G_{[R]}$ is isomorphic to the class group of R. We will prove this claim using Lemma 6.2 which is of independent interest but requires some preliminary definitions:

Suppose for the moment that R is an arbitrary left Noetherian ring. The class of finitely generated projective R-modules, R-**Proj**, is often studied using the group $K_0(R)$. This group is, by definition, the Abelian group generated by the elements of R-**Proj** subject to the relations

$$\langle P \rangle = \langle P_1 \rangle + \langle P_2 \rangle$$

whenever $P \cong P_1 \oplus P_2$ for $P, P_1, P_2 \in R$-**Proj**. Here we use additive notation and write $\langle P \rangle$ for the image of $P \in R$-**Proj** in $K_0(R)$.

Modules $P, Q \in R$-**Proj** are **stably isomorphic** if there is some module $U \in R$-**Proj** such that $P \oplus U \cong Q \oplus U$, or, equivalently, if there is $n \in \mathbb{N}$ such that $P \oplus R^n \cong Q \oplus R^n$. For $P, Q \in R$-**Proj**, $\langle P \rangle = \langle Q \rangle$ in $K_0(R)$ if and only if P and Q are stably isomorphic ([17, Chapter 12], [18]).

A ring R is **left regular** if every finitely generated left R-module has a finite projective resolution. For a left Noetherian ring this means that for every $A \in R$-**Noeth**, there is an exact sequence

$$0 \to P_n \to P_{n-1} \to \ldots \to P_2 \to P_1 \to A \to 0$$

with $P_1, P_2 \ldots, P_n \in R$-**Proj**. For example, rings of finite global dimension and hereditary rings are left regular rings.

Lemma 6.2. *Suppose R is a left Noetherian left regular ring. For all $P, Q \in R$-**Proj** we have*

$$\langle P \rangle = \langle Q \rangle \iff [P] = [Q],$$

that is, P and Q are stably isomorphic if and only if they have isomorphic submodule series.

Proof. We define a map $\Lambda \colon R$-**Noeth** $\to K_0(R)$ as follows. Given a module $A \in R$-**Noeth** with a finite projective resolution as described above, define $\Lambda(A) = \langle P_1 \rangle - \langle P_2 \rangle + \ldots \pm \langle P_n \rangle$. Using Schanuel's Lemma ([20, 3.62, Ex. 3.37]) it is easy to show that Λ is well defined. A simple induction from the Horseshoe Lemma ([20, 6.20]) shows that Λ respects short exact sequences. Thus there is an induced monoid homomorphism $\lambda \colon M(R$-**Noeth**$) \to K_0(R)$ such that $\lambda([A]) = \Lambda(A)$ for all $A \in R$-**Noeth**. In particular, for any $P \in R$-**Proj** we have $\lambda([P]) = \Lambda(P) = \langle P \rangle$. Thus if $P, Q \in R$-**Proj** are such that $[P] = [Q]$, then $\langle P \rangle = \lambda([P]) = \lambda([Q]) = \langle Q \rangle$.

Conversely, if $\langle P \rangle = \langle Q \rangle$, then $P \oplus U \cong Q \oplus U$ for some $U \in R$-**Proj**. From 1.1 we get $[P] = [Q]$. □

Now let R be a Dedekind domain. Then R is Noetherian [19, 1.4.5] and has the following properties:

- **P1** [19, 1.4.5] Every fractional ideal of R is in R-**Proj**. This implies that R is hereditary, has global dimension 1, and hence satisfies the hypothesis of Lemma 6.2.
- **P2** [19, 1.4.11] If I, J are nonzero fractional ideals then $I \oplus J \cong R \oplus IJ$.
- **P3** [19, 1.4.12] Any module $P \in R$-**Proj** is isomorphic to $R^n \oplus I$ for some $n \in \mathbb{Z}^+$ and fractional ideal I. The isomorphism class of I is uniquely determined by P.
- **P4** For $P, Q \in R$-**Proj**, $P \cong Q \iff \langle P \rangle = \langle Q \rangle$. This follows easily from the previous property.

The **class group** of R, $\mathcal{C}(R)$, is the set of isomorphism classes of nonzero fractional ideals of R with operation $+$ defined by $\langle\!\langle I\rangle\!\rangle + \langle\!\langle J\rangle\!\rangle = \langle\!\langle IJ\rangle\!\rangle$. Here we write $\langle\!\langle I\rangle\!\rangle$ for the isomorphism class of the fractional ideal I. The identity element of $\mathcal{C}(R)$ is $\langle\!\langle R\rangle\!\rangle$.

Theorem 6.3. *If R is a Dedekind domain, then $G_{[R]} \cong \mathcal{C}(R)$.*

Proof. For this proof we will consider $G_{[R]}$ to be the set $\{\equiv [R]\}$ with operation $\square_{[R]}$ and identity $[R]$ as described in 2.7(3).

Any nonzero fractional ideal I of R is isomorphic to a nonzero (integral) ideal of R, so $[I] \le [R]$. Further, for any nonzero element x of I, we have $R \cong Rx \subseteq I$, and so $[I] \ge [R]$. Thus $[I] \equiv [R]$, and since elements of $\mathcal{C}(R)$ are isomorphism classes, there is a well defined map $\nu\colon \mathcal{C}(R) \to G_{[R]}$ such that $\nu(\langle\!\langle I\rangle\!\rangle) = [I]$ for all nonzero fractional ideals I. That $\nu\colon \mathcal{C}(R) \to G_{[R]}$ is a group homomorphism follows immediately from the fact that $M(R\text{-}\mathbf{Noeth})$ is strongly separative, 2.7(4) and P2.

From P1, P4 and 6.2 we have, for nonzero fractional ideals I, J,

$$[I] = [J] \iff \langle I\rangle = \langle J\rangle \iff I \cong J \iff \langle\!\langle I\rangle\!\rangle = \langle\!\langle J\rangle\!\rangle.$$

Thus the map ν is injective.

To prove that ν is surjective we need to show that for every $[A] \equiv [R]$ there is a fractional ideal I such that $[A] = [I]$.

Suppose then that $[A] \equiv [R]$ for some $A \in R\text{-}\mathbf{Noeth}$. Then there is an epimorphism from R^n to A for some $n \in \mathbb{N}$. The kernel of this epimorphism is a submodule of R^n so is projective. Without loss of generality we can assume that the kernel is nonzero, hence, using P3, we get the short exact sequence

$$0 \to J \oplus R^m \to R^n \to A \to 0$$

for some nonzero fractional ideal J and $m \in \mathbb{Z}^+$. In $M(R\text{-}\mathbf{Noeth})$ we have $n[R] = m[R] + [J] + [A]$.

The inverse of $\langle\!\langle J\rangle\!\rangle$ in $\mathcal{C}(R)$ is represented by a nonzero fractional ideal I such that $I \oplus J \cong R \oplus R$. In $M(R\text{-}\mathbf{Noeth})$ this implies $[I] + [J] = 2[R]$. Adding $[I]$ to $n[R] = m[R] + [J] + [A]$ yields $n[R] + [I] = (m+2)[R] + [A]$. Now we apply the monoid homomorphism $\phi_{[R]}$ to this equation noting that, since $[A] \equiv [I] \equiv [R]$, we have $\phi_{[R]}([A]) = \phi_{[R]}([I]) = \phi_{[R]}([R]) = 1$. Thus $n+1 = m+2+1$ and so $n[R]+[I] = n[R]+[A]$. Finally we have $n[R] \propto [A]$, so, using the strong separativity of $M(R\text{-}\mathbf{Noeth})$ in the form of 2.3(T3), we can cancel $n[R]$ from this equation to get $[A] = [I]$. $\square$

L. Claborn [7] showed that any Abelian group is the class group of a Dedekind domain, thus we now know that any Abelian group is $G_{[R]}$ for some 1-dimensional Noetherian domain R.

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