California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis   Fall 2003

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Do five of the following seven problems.

Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z;r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a\sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a\sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2003 # 1. **a. (12 points)** Describe and sketch each of the following regions. (Giving reasons for your answers.)

(i) $A = \left\{ z \in \mathbb{C} : \text{Im}\left(\frac{z+1}{z-1}\right) < 0 \right\}$

(ii) $B = \left\{ z \in \mathbb{C} : \text{Re}\left(\frac{z+1}{z-1}\right) < 0 \right\}$

b. (8 points) Find a fractional linear (Möbius) transformation f such that

$$f(i) = -i, \quad f(0) = -1, \quad \text{and} \quad f(-1) = 0.$$

(You may do parts **a** and **b** in either order, and they may or may not be related.)

Fall 2003 # 2. Suppose $f : \Omega \to \mathbb{C}$ is analytic on an open subset Ω of $\mathbb{C}$. For $z = x+iy$ in Ω with x and y real, let $u(x,y) = \text{Re}(f(x+iy))$ and $v(x,y) = \text{Im}(f(x+iy))$

a. State the Cauchy-Riemann equations for u and v and show how they follow from the existence of $f'(z)$.

b. Show that u and v are harmonic on Ω.

c. Find a harmonic conjugate $v(x,y)$ for the function $u(x,y) = 1 + 2x + y^3 - 3x^2y$.

Fall 2003 # 3. Find the Laurent series for $f(z) = \dfrac{1}{(z-1)(z-2)}$ valid in each of the following regions.

a. $A = \{z \in \mathbb{C} : 0 < |z-1| < 1\}$

b. $B = \{z \in \mathbb{C} : 1 < |z-1|\}$

Fall 2003 # 4. Let $f(z) = \dfrac{z^2}{e^z - 1}$.

a. Find all the singularities of f in $\mathbb{C}$ and classify each as a removable singularity, a pole, or an essential singularity. For poles, specify the order of the pole.

b. Evaluate $\int_\gamma f(z)\,dz$ for each of the following paths γ.

(i) the circle of radius 1 centered at 0 traveled once counterclockwise

(ii) the circle of radius 8 centered at 0 traveled once counterclockwise

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Fall 2003 # 5. Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be analytic on all of $\mathbb{C}$.

a. Show that if $\lim_{z \to \infty} |g(z)| = 0$, then $g(z) = 0$ for all z in $\mathbb{C}$.

b. Show that if $\lim_{z \to \infty} |f'''(z)| = 0$, then f must be a polynomial.

Fall 2003 # 6. Evaluate each of the following integrals. Show any curves and explain estimates needed to justify your method.

a. $\displaystyle\int_0^{2\pi} \frac{dt}{4+\sin t}$ **b.** $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$ **c.** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$

Fall 2003 # 7. Consider the problem: Find a function f with

$$f'(z) - f(z) = z \qquad \text{and} \qquad f(0) = 1.$$

Suppose f has a series solution $f(z) = \sum_{n=0}^{\infty} a_n z^n$ valid in some neighborhood of 0.

a. Compute what a_1, a_2, and a_3 would have to be.
b. Find what the series would have to be.
c. Show that the series converges to a solution which is an entire function.

(You may leave the solution as an infinite series if you need to.)

End of Exam