California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis Fall 2004

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Do five of the following seven problems.

Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b) \qquad 2 \cos a \cos b = \cos(a - b) + \cos(a + b)$$

$$2 \sin a \cos b = \sin(a + b) + \sin(a - b) \qquad 2 \cos a \sin b = \sin(a + b) - \sin(a - b)$$

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \qquad \cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2004 # 1. How many values are there for $(1+i)^{2/3}$? Write each in polar form $(re^{i\theta})$ and in rectangular form $(a+bi)$. Sketch their location(s) in the plane.

Fall 2004 # 2. For each of the following real valued functins $u(x,y)$ determine whether it can be the real part of an analytic function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ with $\mathrm{Im}(f(z)) = v(0,0) = 0$. If it can be, find $v(x,y)$. If it cannot, explain how you know that.

a. $u(x,y) = x^3 + 3x^2y - 3xy^2 - y^3$
b. $u(x,y) = 4x^3y + 2xy - 1$

Fall 2004 # 3. Evaluate each of the following integrals.

a. $\displaystyle\int_\gamma \frac{\sin(z^2)}{z^7}\,dz$ γ is the circle of radius 1 centered at the origin and traveled once counterclockwise.

b. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx$ Show any contours and explain estimates needed to justify your method.

Fall 2004 # 4. Let A be the disk $A = \{z \in \mathbb{C} : |z-1| < 1\}$.
Let B be the half-disk $B = \{z \in \mathbb{C} : |z-1| < 1 \text{ and } \mathrm{Im}(z) > 0\}$.
Let C be the half-plane $C = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$
Let D be the quarter-plane $D = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$
Let E be the half-plane $E = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$

a. Find $f : A \to C$ mapping A one-to-one analytically onto C
b. Find $g : B \to E$ mapping B one-to-one analytically onto E

(Hint: How might D figure into this problem?)

Fall 2004 # 5. Let $p(z) = z^{10} - 3z^3 + 1$. Let A be the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$

a. Counting possible multiplicity, how many zeros does the polynomial p have in A?
b. Show that none of the zeros of p in A can have multiplicity larger than 1.

Fall 2004 # 6. Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$ and that it is an isometry in the sense that $|f(z) - f(w)| = |z - w|$ for all z and w in $\mathbb{C}$. Show that there are constants a and b such that $f(z) = az + b$ for all z in $\mathbb{C}$

Fall 2004 # 7. Let $A = \{z \in \mathbb{C} : 1 \le |z| \le 2\}$.
Let C_1 and C_2 be the boundary circles, $C_1 = \{z \in \mathbb{C} : |z| = 1\}$ and $C_2 = \{z \in \mathbb{C} : |z| = 2\}$.
Suppose f is a complex valued function analytic on an open set containing A such that $|f(z)| \le 3$ for all z on C_1 and $|f(z)| \le 12$ for all z on C_2.
Show that $|f(z)| \le 3\,|z|^2$ for all z in A.

End of Exam