California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis   Fall 2005

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z;r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2005 # 1. **a.** Suppose f is a complex valued function analytic on an open set A in $\mathbb{C}$. Give a statement of the Cauchy-Riemann equations for f and show how they follow from the assumption of the existence of the complex derivative.

b. Let D be the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose $f : D \to \mathbb{C}$ is analytic on D and that $\text{Re}(f(z)) = \text{Im}(f(z))$ for all z in D. Show that f must be constant on D.

Fall 2005 # 2. Evaluate the integral $\displaystyle\int_\gamma \frac{e^{2z}\,dz}{(z-2)(z-4)}$ Where γ is

(a) the circle $\{z : |z| = 1\}$ traveled once counterclockwise.
(b) the circle $\{z : |z| = 3\}$ traveled once counterclockwise.
(c) the circle $\{z : |z| = 5\}$ traveled once counterclockwise.
(d) The polygonal path obtained by following straight line segments from $3i$ to $6-3i$ to $6+3i$ to $-3i$ and back to $3i$ in that order

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Fall 2005 # 3. Evaluate each of the following integrals

a. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{z^6+1}\,dz$ **b.** $\displaystyle\int_\gamma z^3\cos(1/z)\,dz$ **c.** $\displaystyle\int_0^{2\pi} \frac{1}{10-6\cos\theta}\,d\theta$

In **(a)** show any contours and explain any estimates needed to justify your method.
In **(b)** the curve γ is the unit circle traveled once in the counterclockwise direction.

Fall 2005 # 4. Let $f(z) = z/(e^z - 1)$.

a. Find all singularities of f in $\mathbb{C}$ and classify each as removable, a pole (specify the order), or essential.

b. One definition of the Bernoulli numbbers B_n relates them to the coefficents of an expansion of f by

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

Explain why f has a series expansion like this.

c. Find B_0, B_1, and B_2.

d. What is the radius of convergence of the series in part **(b)** ?

Fall 2005 # 5. Find the Laurent series for $f(z) = \dfrac{1}{z(z-1)}$ valid in each of the following regions

a. $\{z \in \mathbb{C} : 0 < |z| < 1\}$ **b.** $\{z \in \mathbb{C} : 1 < |z|\}$ **c.** $\{z \in \mathbb{C} : 0 < |z-1| < 1\}$

Fall 2005 # 6. Let $D = \{z \in \mathbb{C} : |z| < 1\}$

a. Find a conformal map $f : D \to D$ of D onto itself such that $f(1/2) = 0$.
b. Find a conformal map $g : D \to D$ of D onto itself such that $g(1/2) = 1/3$.

Fall 2005 # 7. Show that the zeros of $f(z) = z^4 + 3iz^2 + 3$ lie in the disk $\{z \in \mathbb{C} : |z| \leq \sqrt{4}\}$.

End of Exam