California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis Fall 2006

Chang*, Cooper, Hoffman, Krebs

Do five of the following eight problems.
If you attempt more than 5, the best 5 will be used.

Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\text{Log}\, z$ denotes the principal branch of $\log z$.
$\text{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2006 # 1. Sketch each of the following regions in the complex plane

a. $\{z \in \mathbb{C} : 0 \le \arg(z) \le \pi/4 \text{ and } 1 \le |\,z\,| \le 2\}$
b. $\{z \in \mathbb{C} : \mathrm{Re}(z^2) \ge 0\}$
c. $\{z \in \mathbb{C} : \mathrm{Re}(1/z) = 1/2\}$
d. $\{z \in \mathbb{C} : z^3 = -8\}$

Fall 2006 # 2. For each of he following functions $u : \mathbb{R}^2 \to \mathbb{R}$, determine whether there is a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $f(z) = f(x+iy) = u(x,y)+iv(x,y)$ is analytic with $f(0) = 1 + 2i$. If there is such a function v, find one. If there is not, explain how you know there is not.

a. $u(x,y) = y + e^x \cos y$
b. $u(x,y) = y^2 + e^x \cos y$

Fall 2006 # 3. Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic on the whole plane and that $u(x,y) < 0$ for all (x,y) in $\mathbb{R}^2$. Show that u must be constant.

Fall 2006 # 4. Suppose $f : U \to \mathbb{C}$ is analytic on an open set U contining the closed unit disk $D = \{z \in \mathbb{C} : |\,z\,| \le 1\}$, and that $f(0) = 0$ and $|\,f(z)\,| \le 1$ for all z in D.

Let $h(z) = \dfrac{f(z) + f(-z)}{2}$

a. Show that h has a zero of order at least 2 at 0.
b. Show that $|\,h(z)\,| \le |\,z\,|^2$ for all z in D.

Fall 2006 # 5. Evaluate each of the following integrals. Show any curves and explain any estimates needed to justify your method.

$$\textbf{a.} \quad \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx \qquad \textbf{b.} \quad \int_0^{2\pi} \frac{1}{2+\cos\theta}\,d\theta$$

Fall 2006 # 6. The parts of this problem really do not have much to do with each other except that they are both about series expansions.

a. Find the Laurent series for $\dfrac{1}{(z-2)(z-4)}$ valid for $|\,z-2\,| > 2$.

b. Suppose $f(z)$ is analytic for $|\,z\,| < 2$. For real t, let $F(t) = f(e^{it})$. Show that for each t,

$$F(t) = \sum_{k=0}^{\infty} a_k e^{ikt} \qquad \text{where } a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)e^{-ik\theta}\,d\theta.$$

Fall 2006 # 7. Find the image of the disk $D = \{z \in \mathbb{C} : |\,z\,| < 1\}$ under the mapping $z \mapsto w = \dfrac{1}{z-1}$. Justify your answer. (Suggestion: What happens to the boundary circle?)

Fall 2006 # 8. Counting possible multiplicity, how many zeros does the function $f(z) = z^6 - 5z^4 + 2z^2 - 1$ have in the disk $D = \{z \in \mathbb{C} : |\,z\,| < 1\}$? Justify your answer.

End of Exam