California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis   Fall 2011

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\text{Log}\, z$ denotes the principal branch of $\log z$.
$\text{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2011 # 1. Sketch (and describe as appropriately helpful) each of the following sets in $\mathbb{C}$.

a. $A = \{z \in \mathbb{C} : \mathrm{Im}(z) = \mathrm{Re}(z)\}$
b. $B = \{z \in \mathbb{C} : \mathrm{Im}(z^2) = \mathrm{Re}(z^2)\}$
c. $C = \{z \in \mathbb{C} : \mathrm{Re}(z^2 - 1) \geq 0\}$
d. $C = \{z \in \mathbb{C} : |\, z \,| \leq \arg z \text{ and } 0 \leq \arg z \leq \pi\}$

Fall 2011 # 2. For each of the following, classify the singularity at the indicated point as removable, a pole (state the order of each pole), or essential and find the residue at that point.

a. $f(z) = \dfrac{z}{z^2 - 1}, \quad z_0 = 1$ **a.** $\dfrac{e^z - 1}{\sin z}, \quad z_0 = 0$

c. $f(z) = z^n e^{1/z}, \quad z_0 = 0$ **d.** $f(z) = \dfrac{e^z - 1}{z^2}, \quad z_0 = 0$

(In part **(c)** n is a positive integer and the answer should be in terms of n.

Fall 2011 # 3. Use complex variable methods to evaluate each of the following integrals. Sketch any contours and discuss estimates needed to justify your methods.

$$\textbf{a}. \quad \int_0^\infty \frac{1}{1 + x^4}\, dx \qquad \textbf{b}. \quad \int_0^{2\pi} \frac{1}{5 + 3\cos\theta}\, d\theta$$

Fall 2011 # 4. **a.** Prove the Cauchy's Inequality: If f is analytic on an open set A which contains the circle $\gamma = \{z \in \mathbb{C} : |\, z - z_o \,| = R\}$ and its interior and $|\, f(z) \,| \leq M$ for all z on γ, then

$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$$

for $k = 0, 1, 2, 3, \ldots$.

b. State and prove Liouville's Theorem about bounded entire functions using Cauchy's Inequality.

Fall 2011 # 5. (Note: You do not need to know anything about Fourier series other than the definitions given here to do this problem. It really is a complex analysis problem)

If $F(\vartheta)$ is a 2π-periodic function of ϑ, the Fourier coefficients of F are defined for integer n by $\hat{F}(n) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} F(\vartheta) e^{-in\vartheta}\, d\vartheta$. Suppose that $r > 1$ and that $f(z)$ is a complex valued function analytic on the disk $D = \{z \in \mathbb{C} : |\, z \,| < r\}$. Let $F(\vartheta) = f(e^{i\vartheta})$.

(a) Show that $\hat{F}(n) = 0$ for $n < 0$, and $\hat{F}(n) = \dfrac{f^{(n)}(0)}{n!}$ for $n \geq 0$.

(b) Show that the Fourier series $\displaystyle\sum_{-\infty}^{\infty} \hat{F}(n) e^{in\vartheta}$ converges to $F(\vartheta)$ for each ϑ with $-\pi < \vartheta \leq \pi$.

Fall 2011 # 6. Show that $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ converges to an analytic function on the set $A = \{ z \in \mathbb{C} \mid |z| < 1\}$.

Fall 2011 # 7. Let D be the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$ and Q be the open first quadrant, i.e. $Q = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$.

a. (15 pts): Find a function f analytic on Q mapping Q one-to-one onto D with $f(1+i) = 0$.

b. (5 pts): Can the mapping requested in part (a) be accomplished by a single fractional linear (Mobius) transformation? Why or why not?

End of Exam