California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis Fall 2013

Chang, Gutarts*, Hoffman, Wang

Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\text{Log}\, z$ denotes the principal branch of $\log z$.
$\text{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z;r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

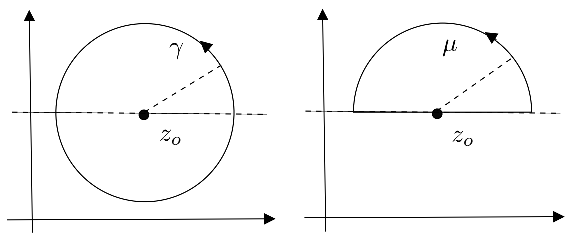
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Fall 2013 # 1. Suppose $u(x, y)$ and $v(x, y)$ are real valued harmonic functions on an open set Ω in the plane and that v is a harmonic conjugate for u on Ω. Show that the product uv is also harmonic on Ω.

Fall 2013 # 2. Let γ be a circle of radius r centered at $z_0 \in \mathbb{C}$, travelled counterclockwise, and let μ be the top half of the same circle. (See figure.) (μ is just the semicircle – not a closed curve.)For each integer n, evaluate the integrals

$$\int_\gamma (z - z_0)^n \, dz \qquad \text{and} \qquad \int_\mu (z - z_0)^n \, dz.$$



(Your answers will probably depend on n. Make sure you treat all integer n.)

Fall 2013 # 3. For each of the following find the Taylor series centered at 0 using methods specified. For each, give the radius of convergence.

a. $f(z) = e^{-z}$ (Directly from the definition by computing the Taylor coefficients)

b. $g(z) = \dfrac{z}{(1 - z)^2}$ (Start with the geometric series for $1/(1 - z)$.)

c. $h(z) = \log(1 + z)$ (Your choice of method).

Make sure you explain your methods.

Fall 2013 # 4. Suppose the power series $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence 2. Call the value of the sum $f(z)$ and let $g(z) = f(z)/(1 - z)$.

a. Explain how you know $g(z)$ has a series representation of the form $g(z) = \sum_{n=0}^{\infty} b_n z^n$ valid at least for $|z| < 1$.

b. For each $n = 0, 1, 2, 3$, find b_n in terms of a_0, a_1, a_2, a_3, . . .

Fall 2013 # 5. Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic on all of $\mathbb{C}$, and $\left| f^{(3)}(z) \right| < \mathrm{M}$ for all z in $\mathbb{C}$. Show that f is a polynomial. What can you say about the degree of f.

Fall 2013 # 6. Let γ be the circle $\{z \in \mathbb{C} : |\, z \,| = 1\}$. Suppose f is a function analytic on an open set containing γ and its interior and that $|\, f(z) \,| < 1$ for each z on γ. Show that f has exactly one fixed point inside γ. (That is, there is exactly one z in the open unit disk with $f(z) = z$.)

Fall 2013 # 7. **a.** Suppose $f : A \to \mathbb{C}$ is analytic on an open set A containing the closed half plane $H = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$ and that there is a finite constant M with $|\, f(z)\,| \leq M$ for all z in H.

i. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{f(x)}{x^2+1} = \pi f(i)$.

ii. Show that if w_o is a point in $\mathbb{C}$ with $\mathrm{Im}(w_o) > 0$, then

$$\int_{-\infty}^{\infty} \frac{f(x)}{x^2 - (2\,\mathrm{Re}(w_o))\,x + |\,w_o\,|^2} = \frac{\pi f(w_o)}{\mathrm{Im}(w_o)}$$

. (Suggestion: the denominator is easy to factor. Compare to part (i).)

Sketch any curves and discuss estimates needed to justify your method. You may do parts (i) and (ii) in either order.

b. Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{5 + 4\cos\theta}\, d\theta$

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End of Exam