California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis   Spring 2003

Chang, Hoffman*, Katz

Do five of the following eight problems.

Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\text{Log}\, z$ denotes the principal branch of $\log z$.
$\text{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

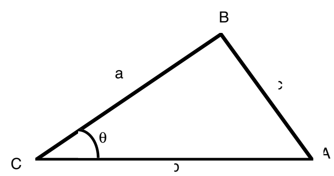
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2003 # 1. **a.** Show that if z and w are in $\mathbb{C}$, then

$$|z-w|^2 = |z|^2 + |w|^2 - 2\,\mathrm{Re}(z\,\overline{w}).$$

b. Use the result of part **a** to obtain the "law of cosines" for triangles.



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Spring 2003 # 2. Let $f(z) = \dfrac{1}{(z-1)(z-2)}$ and $g(z) = \dfrac{z}{(z-1)(z-2)}$.

Let γ be the circle of radius 4 centered at the origin and traveled once in the counter-clockwise direction. Let $A = \{z \in \mathbb{C} : |z| > 3\}$.

- **a.** Evaluate $\int_\gamma f(z)\,dz$
- **b.** Evaluate $\int_\gamma g(z)\,dz$.
- **c.** Show there is no analytic function $G : A \to \mathbb{C}$ such that $G'(z) = g(z)$ for all z in A.
- **d.** Show there is an analytic function $F : A \to \mathbb{C}$ such that $F'(z) = f(z)$ for all z in A. (You do not need to actually display such a function, but if you can, that would be one way to answer the problem.)

Spring 2003 # 3. Let $A = \{z \in \mathbb{C} : 0 < |z| < 1\}$, and suppose that $f : A \to \mathbb{C}$ is analytic on A. For $0 < r < 1$, define $m(r)$ by $m(r) = \displaystyle\int_0^{2\pi} f(re^{i\theta})\,d\theta$.

Show that $m(r)$ is constant on the inteval $0 < r < 1$.

Spring 2003 # 4. Suppose $\{p_k\}_{k=1}^{\infty}$ is a sequence of polynomials and that $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $p_k(z) \to f(z)$ as $k \to \infty$ with the convergence uniform on each closed disk $D_R = \{z \in \mathbb{C} : |z| \le R\}$.

- **a.** Show that if there is a finite constant M such degree$(p_k) \le M$ for each k, then f must be a polynomial.
- **b.** Give an example to show that if there is no such constant M, then f might not be a polynomial.

Spring 2003 # 5. How may solutions are there (counting possible multiplicity) to the equation $e^z = 4z + 1$ in the disk $D = \{z \in \mathbb{C} : |z| \le 1\}$?

Spring 2003 # 6. For each of the following regions, give reasons why there can be no analytic function from $\mathbb{C}$ one-to-one onto that region. (Do not use Picard's Theorem.)

a. $A = \{z \in \mathbb{C} : |z| < 1\}$
b. $B = \{z \in \mathbb{C} : \text{Re}(z) > 1\}$
c. $C = \mathbb{C} \setminus \{\text{the non-positive real axis}\} = \mathbb{C} \setminus \{z \in \mathbb{C} : \text{Im}(z) = 0 \text{ and } \text{re}(z) \leq 0\}$
d. $D = \mathbb{C} \setminus \{0\}$

Spring 2003 # 7. Evaluate each of the following integrals. Show any contours and explain any estimates needed to justify your method.

$$\textbf{a.} \quad \int_0^\infty \frac{x^2}{x^4 + 16}\, dx \qquad\qquad \textbf{b.} \quad \int_0^{2\pi} \frac{1}{2 + \cos\theta}\, d\theta$$

Spring 2003 # 8. For each positive real number t, define $g_t(z)$ by $g_t(z) = (1 + tz)^t$.

a. Why doe $g_t(z)$ have an expansion of the form $g_t(z) = \sum_{n=0}^{\infty} P_n(t) z^n$ valid for z at least in some disk around $z = 0$?
b. Find $P_0(t)$, $P_1(t)$, and $P_2(t)$.
c. What is the radius of convergence of the series in **a**? (Be sure to consider the possibilties that t is an integer and that t is not an integer.)

End of Exam