California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis   Spring 2005

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2005 # 1. Sketch each of the following regions in the complex plane.

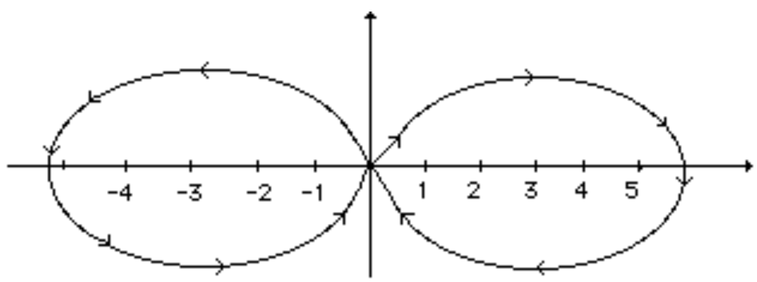
(a) $\{z \in \mathbb{C} : \mathrm{Re}\left(z^2\right) > 1\}$
(b) $\{z \in \mathbb{C} : \mathrm{Im}\left(z^2\right) > 1\}$
(c) $\{z \in \mathbb{C} : |2z - 1| < |2 - z|\}$

Spring 2005 # 2. For each of the following real valued functions $u(x, y)$ determine whether it can be the real part of an analytic function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ with $f(0) = 3i$. If it can be, find $v(x, y)$. If it cannot, explain how you know that.

a. $u(x, y) = x^2 + y^2$
b. $u(x, y) = x^3 - 3xy^2 + y$

Spring 2005 # 3. Evaluate the integral $\displaystyle\int_\gamma \frac{e^z\,dz}{z^2 - 2z - 15}$ Where γ is

(a) the circle of radius $\{z : |z| = 2\}$ traveled once counterclockwise.
(b) the circle of radius $\{z : |z| = 4\}$ traveled once counterclockwise.
(c) the circle of radius $\{z : |z| = 6\}$ traveled once counterclockwise.
(d) The "figure eight" shown in the sketch.



The curve γ for Problem 5d

Spring 2005 # 4. Let $f(z) = \dfrac{z^2 - 1}{\cos(\pi z/2)}$.

a. Find all the singularities of f in $\mathbb{C}$ and classify each as removable, a pole, or essential. For poles, give the order.
b. Explain how you know that $f(z)$ has a series expansion of the form $\sum_{n=0}^{\infty} a_n z^n$ valid for z near 0, and compute a_0, a_1, and a_2.
c. Find the radius of convergence of the series in part **(b)** and explain how you know.

Spring 2005 # 5. Evaluate each of the following three quantities. Show any contours and explain any estimates needed to justify your method. The curve γ for part **(c)** is the circle of radius 1 centered at the origin and travelled once in the counterclockwise direction.

a. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1}\,dx$ **b.** $\displaystyle\int_0^{2\pi} \frac{1}{3 + 2\cos\vartheta}\,d\vartheta$ **c.** $\displaystyle\int_\gamma z^5 \cos(1/z)\,dz$

Spring 2005 # 6. Let C_1 and C_2 be the circles (each with the origin deleted)

$$C_1 = \{z \in \mathbb{C} : |z-1| = 1\} \setminus \{0\} \quad , \quad C_2 = \{z \in \mathbb{C} : |z-2| = 2\} \setminus \{0\}$$

Let A be the region between these circles: $A = \{z \in \mathbb{C} : |z-1| > 1 \text{ and } |z-2| < 2\}$. Let $f(z) = 4/z$

a. Describe and sketch the image sets $f(C_1)$, $f(C_2)$, and $f(A)$ justifying your answers.

b. With $x = \text{Re}(z)$ and $y = \text{Im}(z)$, find a real valued function $\phi(x, y)$ which satisfies all the following conditions

i. ϕ is harmonic on A

ii. $\phi(x, y) = 1$ for $z = x + iy \in C_2$

iii. $\phi(x, y) = 2$ for $z = x + iy \in C_1$.

Spring 2005 # 7. Let γ be the circle $\{z \in \mathbb{C} : |z| = 1\}$.

Suppose f is a function analytic on an open set contining γ and its interior and that $|f(z)| < 1$ for each z on γ.

Show that f has exactly one fixed point inside γ. (That is, there is exactly one z in the open unit disk with $f(z) = z$.)

End of Exam