California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis Spring 2011

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\text{Log}\, z$ denotes the principal branch of $\log z$.
$\text{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

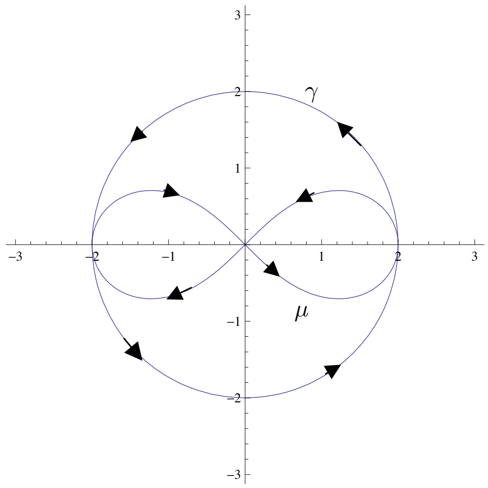
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2011 # 1. Sketch (and describe as appropriately helpful) each of the following sets in $\mathbb{C}$.

a. $A = \{z \in \mathbb{C} : \mathrm{Re}(e^z) \leq 0\}$
b. $B = \{z \in \mathbb{C} : \mathrm{Re}(z^2) \leq 0\}$
c. $C = \{z \in \mathbb{C} : |\, z - 3\,| \leq |\, z + 1\,|\}$

Spring 2011 # 2. Consider the function $f(z) = e^z/(z^2 - 1)$ and the curves γ and μ in the sketch. γ is the circle of radius 2 centered at the origin and travelled once on the counterclockwise direction. μ is a lemniscate "figure eight" travelled as indicated by the arrows.



a. Evaluate $\displaystyle\int_\gamma f(z)\,dz$

b. Evaluate $\displaystyle\int_\mu f(z)\,dz$

c. Can the curve γ be continuously deformed to the curve μ without leaving the region $\mathbb{C} \setminus \{-1, 1\}$?
Give reasons for your answer.

Spring 2011 # 3. Let $D = \{z \in \mathbb{C} : |\, z\,| < 1\}$ be the open unit disk in the complex plane.

a. Suppose that f and g are complex valued analytic functions on D such that $\mathrm{Re}(f(z)) = \mathrm{Re}(g(z))$ for all z in D. Show that $f - g$ is constant on D.
b. Suppose that $f : D \to \mathbb{C}$ is analytic on D with $f(0) = 0$ and $\mathrm{Re}(f(x+iy)) = x^3 - 3xy^2 + y$ for all $z = x + iy$ in D. Find a formula for $\mathrm{Im}(f(x + iy))$

Spring 2011 # 4. Use complex variable methods to evaluate each of the following integrals. Sketch any contours and discuss estimates needed to justify your methods.

a. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4}\,dx$ **b**. $\displaystyle\int_{-\pi}^{\pi} \frac{1}{3 + 2\cos\theta}\,d\theta$ **c**. $\displaystyle\int_\gamma z^5 e^{i/z^2}\,dz$

In part **c**, γ is the circle of radius 1 centered at 0 travelled once in the counterclockwise direction.

Spring 2011 # 5. Suppose $u(x, y)$ is a real valued function which is harmonic on the whole plane such that $|\, u(x, y)\,| \leq 17$ for every $z = x + iy$ in $\mathbb{C}$. Use facts from complex analysis to show that u must be constant.

Spring 2011 # 6. Let $f(z) = e^z - 3z$ and D be the open unit disk $D = \{z \in \mathbb{C} : |\, z \,| < 1\}$.

a. Counting each zero with its multiplicity (order), how many zeros does f have in D?

b. Can any of the zeros of f in D have multiplicity (order) larger than 1? (Justify your answer.)

(If part **b** seems a little strange, do not worry about it, just go ahead and answer it.)

Spring 2011 # 7. Let $f(z) = \dfrac{1}{(z-1)(z-2)}$.

a. Find the Laurent series for f in the region $\{z \in \mathbb{C} : 1 < |\, z \,| < 2\}$.

b. Find the Laurent series for f in the region $\{z \in \mathbb{C} : 0 < |\, z - 1 \,| < 1\}$

c. Find the residue of f at 0 and the residue of f at 1.

End of Exam