California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis Spring 2015

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\mathrm{Log}\, z$ denotes the principal branch of $\log z$.
$\mathrm{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2015 # 1. Consider the fractional linear (Möbius) transformation

$$w = f(z) = \frac{4z-4}{z-2}.$$

Describe and sketch the image set $f(A)$ if A is the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

Spring 2015 # 2. Evaluate each of the following integrals

$$\textbf{a.} \int_\gamma \frac{\overline{z}}{z-4}\,dz \qquad \textbf{b.} \int_0^{2\pi} \frac{d\theta}{3+\sin\theta} \qquad \textbf{c.} \int_0^\infty \frac{dx}{(x^2+1)^2}$$

In **a**, γ is the circle of radius 3 centered at 0 travelled once counterclockwise. (Read part **a** carefully!)

In **c**, show the integral exists, sketch any contours, and discuss any estimates needed to justify your method.

Spring 2015 # 3. **a.** State Liouville's theorem.

b. Is there an entire function f such that $f(0) = 0$ and $f(z) = 1$ whenever $|z| > 1$? (Justify your answer.)

Spring 2015 # 4. Consider the function $f(z) = \dfrac{1}{(z-1)(z-2)}$.

- **a.** Write the Laurent series expansion of $f(z)$ converging in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.
- **b.** Write the Laurent series expansion of $f(z)$ converging in the punctured disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$.

Spring 2015 # 5. For **two** of the following three functions find the isolated singularities in $\mathbb{C}$. Determine whether each is removable, essential, or a pole. Determine the order of each pole and find the principal part at each pole.

$$\textbf{a.} \quad \frac{z}{(z^2-1)^2} \qquad \textbf{b.} \quad \exp\left(\frac{1}{z^2+1}\right) \qquad \textbf{c.} \quad z^2\sin(1/z)$$

Spring 2015 # 6. Give a statement of Rouche's theorem, and use it to find the number of zeros of the polynomial $p(z) = z^6 + (1+i)z + 1$ in the annulus $\{z \in \mathbb{C} : 1/2 < |z| < 5/4\}$.

Spring 2015 # 7. Let f and g be complex valued functions each analytic on an open set containing the closed disk $B = \{z \in \mathbb{C} : |z| \le 1\}$ and suppose that neither has any zeros in B. Show that if $|f(z)| = |g(z)|$ for all z with $|z| = 1$, then there is a real number θ such that $f(z) = e^{i\theta}g(z)$ for all z in B.

(Suggestion: It might be helpful to consider f/g and g/f)

End of Exam