California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Complex Analysis   Spring 2018

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\operatorname{Log} z$ denotes the principal branch of $\log z$.
$\operatorname{Arg} z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Spring 2018 # 1. For each of the following real valued functions of two real variables $u(x, y)$, decide whether there is a real valued function $v(x, y)$ such that $f(z) = f(x+iy) = u(x, y)+iv(x, y)$ is analytic. If yes, find such a v. If not, explain how you know there is no such function.

a. $u(x, y) = \cos y - xy$ **b.** $u(x, y) = e^{3x}\cos(3y) - xy$

Spring 2018 # 2. **a.** Give a statement of the maximum modulus theorem.
b. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be a continuous function on a closed bounded region R in $\mathbb{C}$ which is analytic and not constant on the interior of R. Show that The real valued function $u(x, y)$ reaches its maximum on the boundary of R. Suggestion: Consider $e^{f(z)}$.

Spring 2018 # 3. Find the Laurent series expansions for $f(z) = \dfrac{1}{z^2(z^2 - 9)}$ around $z_o = 0$ valid in each of the following regions

a. $A = \{z \in \mathbb{C} : 0 < |\, z\,| < 3\}$ **b.** $B = \{z \in \mathbb{C} : |\, z\,| > 3\}$

c. Find the residue of f at $z_o = 0$.

Spring 2018 # 4. Let $f(z) = \dfrac{z^2 e^{1/z}}{z+1}$.
Find all singularities of f in $\mathbb{C}$.
Classify each as removable, a pole (of what order), or essential.
Find the residue of f at each.

Spring 2018 # 5. Use complex variable methods to evaluate **two** of the following integrals. Show the integrals exist, sketch any contours and discuss any estimates needed to justify your method. In **c**, a is a real constant with $a > 1$.

a. $\displaystyle\int_0^\infty \frac{x^2}{x^4 + x^2 + 1}\,dx$ **b.** $\displaystyle\int_{-\infty}^\infty \frac{1 + \cos x}{x^2 + 1}\,dx$ **c.** $\displaystyle\int_0^{2\pi} \frac{1}{a + \cos\theta}\,d\theta$

Spring 2017 # 6. Let $D = \{z \in \mathbb{C} : |\, z\,| < 1\}$ be the open unit disk and $H = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ be open lower half plane. Determine if there exists a conformal mapping of D one-to-one onto H. If not, explain why not with appropriate reasons. If yes, give a full explanation by either displaying such a function and/or by stating and applying appropriate theorem or theorems.

Spring 2018 # 7. **a.** Determine the number of zeros, counting multiplicity, of the polynomial $f(x) = z^5 - 6z^2 + 2z + 1$ in the annulus $A = \{z \in \mathbb{C} : 1 \le |\, z\,| \le 2\}$. Show your work.
b. Show that if n is an integer with $n > 2$, then all the solutions of $z^n - (z^2 + z + 1)/4 = 0$ lie inside the unit circle, $|\, z\,| = 1$.

End of Exam