California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Complex Analysis Winter 2002

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Do five of the following seven problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\text{Re}(z)$ denotes the real part of the complex number z.
$\text{Im}(z)$ denotes the imaginary part of the complex number z.
$\overline{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z.
$\text{Log}\, z$ denotes the principal branch of $\log z$.
$\text{Arg}\, z$ denotes the principal branch of $\arg z$.
$D(z; r)$ is the open disk with center z and radius r.
A *domain* is an open connected subset of $\mathbb{C}$.

MISCELLANEOUS FACTS

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a) \qquad \cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)$$

Winter 2002 # 1. Describe and sketch each of the following sets

a. $A = \left\{ z \in \mathbb{C} : \mathrm{Re}\left(\frac{1}{z}\right) > \frac{1}{2} \right\}$

b. $B = \left\{ z \in \mathbb{C} : \left| \frac{z-2}{z+1} \right| > 2 \right\}$

Winter 2002 # 2. For z in $\mathbb{C}$, let $z = x + iy$ with x and y real. For each of the following real valued functions $u(x, y)$, determine whether there is a real valued function $v(x, y)$ such that the function $f(z) = u(x, y) + iv(x, y)$ is analytic and $f(0) = i$. If there is such a function v, find one and explain how you know that f is analytic. If there is not, explain how you know that there is not.

a. $u(x, y) = (x + 1)y$

b. $u(x, y) = (x + y)y$

Winter 2002 # 3. Let $f(z) = \dfrac{z^2 - 1}{\sin \pi z}$

a. Find all singularities of f in $\mathcal{C}$ and classify each as a pole (specifying the order), essential, removable, or other.

b. Explain why $f(z)$ has a series expansion of the form $\sum_{k=-\infty}^{\infty} c_k z^k$ valid for z near 0. Which, if any, of the coefficients c_k for $k < 0$ are not equal to 0?

c. Find c_{-1}, c_0, and c_1

d. What is the region of validity for the expansion discussed in part **b**?

e. Find $\displaystyle\int_\gamma f(z)\, dz$ where γ is the circle of radius 1 centered at the origin and travelled once counterclockwise.

Winter 2002 # 4. Evaluate the following integrals using complex variable methods. Show any cirves and explain any estimates needed to justify your method.

$$\textbf{a.} \int_0^{2\pi} \frac{d\vartheta}{2 + \cos\vartheta} \quad ; \quad \textbf{b.} \int_{-\infty}^{+\infty} \frac{dx}{x^2 - 4x + 5} \quad ; \quad \textbf{c.} \int_0^{\infty} \frac{\sqrt{x}}{1 + x^2}\, dx$$

Winter 2002 # 5. Let p be a polynomial with $p(0) = 0$.

a. Evaluate $\displaystyle\int_{-\pi}^{\pi} (1 - p(e^{i\theta}))\, d\theta$

b. Show that there is at least one real number θ with $\left|1 - p(e^{i\theta})\right| \geq 1$

Winter 2002 # 6. Let m and n be integers with $m > n > 0$. Let $q(z)$ and $p(z)$ be polynomials of degree m and n

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \qquad \text{and} \qquad q(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_m$$

Let γ_R be the circle of radius R centered at 0 and travelled once counterclockwise. Show that

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma_R} \frac{p(z)}{q(z)}\, dz = \begin{cases} \dfrac{a_0}{b_0}, & \text{if } m = n+1 \\ 0, & \text{if } m - n > 1 \end{cases}$$

Winter 2002 # 7. Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic on all of $\mathbb{C}$ and that there is a polynomial p of degree n and a point z_o such that $|f(z)| \le |p(z)|$ for all z with $|z| \ge |z_o|$.

Prove that f must be a polynomial of degree no more than n.

End of Exam