AVOIDING PATTERNS OF LENGTH THREE IN COMPOSITIONS AND MULTISET PERMUTATIONS

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Abstract

We find generating functions for the number of compositions avoiding a single pattern or a pair of patterns of length three on the alphabet $\{1, 2\}$ and determine which of them are Wilf-equivalent on compositions. We also derive the number of permutations of a multiset which avoid these same patterns and determine the Wilf-equivalence of these patterns on permutations of multisets.

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1. Introduction

Pattern avoidance was first studied for $\mathfrak{S}_n$, the set of permutations of $[n] = \{1, 2, \ldots, n\}$, avoiding a pattern $\tau \in \mathfrak{S}_3$. Knuth [5] found that, for any $\tau \in \mathfrak{S}_3$, the number of permutations of $[n]$ avoiding τ is givien by the nth Catalan number. Later, Simion and Schmidt [8] determined $|\mathfrak{S}_n(T)|$, the number of permutations of $[n]$ simultaneously avoiding any given set of patterns $T \subseteq \mathfrak{S}_3$. Burstein [1] extended this to words of length n on the alphabet $[k] = \{1, \ldots, k\}$, determining the number of words that avoid a set of patterns $T \subseteq \mathfrak{S}_3$. Burstein and Mansour [2] considered forbidden patterns with repeated letters and we will use techniques similar to the ones used in their paper. Recently, pattern avoidance has been studied for compositions. Heubach and Mansour [4] counted the number of times a pattern τ of length 2 occurs in compositions, and determined the number of compositions avoiding such a pattern. Most recently, Savage and Wilf [6] considered pattern avoidance in compositions for a single pattern $\tau \in \mathfrak{S}_3$, and showed that the number of compositions of n with parts in $\mathbb{N}$ avoiding $\tau \in \mathfrak{S}_3$ is independent of τ.

Savage and Wilf posed some open questions, one of which asked about pattern avoidance in compositions where the patterns are not themselves permutations, i.e., the pattern has repeated letters. We will answer this question for all such patterns of length 3, and also consider pattern avoidance for pairs of such patterns. We will derive generating functions and determine which patterns or sets of patterns are avoided equally often.

2. Preliminaries

Let $\mathbb{N}$ be the set of all positive integers, and let A be any ordered finite (or infinite) set of positive integers, say $A = \{a_1, a_2, \dots, a_d\}$, where $a_1 < a_2 < a_3 < \cdots < a_d$. For ease of notation, "ordered set" will always refer to a set whose elements are listed in increasing order.

A *composition* $\sigma = \sigma_1\sigma_2\dots\sigma_m$ of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. The number of *summands* or *letters*, namely m, is called the number of *parts* of the composition. For any ordered set $A = \{a_1, a_2, \dots, a_k\} \subseteq \mathbb{N}$, we denote the set of all compositions of n with parts in A (with m parts in A) by C_n^A ($C_{n;m}^A$).

To define a pattern, we utilize the notion of words. Let $[k]^n$ denote the set of all words of length n over the (totally ordered) alphabet $[k] = \{1, 2, \dots, k\}$. We call these words *k-ary words of length n*. A *pattern* τ is a word in $[\ell]^m$ that contains each letter from $[\ell]$, possibly with repetitions. We say that the composition $\sigma \in C_n^A$ (resp., $\sigma \in C_{n;m}^A$) *contains* a pattern τ, if σ contains a subsequence whose elements are order-isomorphic to τ. Otherwise, we say that σ *avoids* τ and write $\sigma \in C_n^A(\tau)$ (resp., $\sigma \in C_{n;m}^A(\tau)$). Moreover, if T is a set of patterns on $[k]^n$, then $C_n^A(T)$ (resp., $C_{n;m}^A(T)$) denotes the set all compositions in C_n^A (resp., $\sigma \in C_{n;m}^A$) that avoid all patterns from T simultaneously.

For a given set of patterns T and an ordered finite or infinite set A of positive integers, we define $|C_{n;0}^A(T)| = 1$ for all $n \geq 0$ and $|C_{n;m}^A(T)| = 0$ for $n < 0$ or $m < 0$. We denote the generating function of the number for T-avoiding compositions in $C_{n;m}^A$ by $C_T^A(x; m)$; that is,

$$C_T^A(x; m) = \sum_{n \geq 0} |C_{n;m}^A(T)| x^n.$$

Furthermore, we denote the ordinary generating function for $C_T^A(x; m)$ by $C_T^A(x, y)$; that is,

$$C_T^A(x, y) = \sum_{m \geq 0} C_T^A(x; m) y^m.$$

For ease of notation, we denote the ordinary generating function $C_T^A(x, 1)$ by $C_T^A(x)$; that is,

$$C_T^A(x) = \sum_{n \geq 0} |C_n^A(T)| x^n.$$

We say that two sets of patterns T_1 and T_2 belong to the same *cardinality class* or are *Wilf-equivalent* if for all values of A, m and n, we have $|C_{n;m}^A(T_1)| = |C_{n;m}^A(T_2)|$. It is easy to see that for each $\tau \in [\ell]^k$, the *reversal map* defined by $r : \tau_i \mapsto \tau_{k+1-i}$ produces a pattern that is Wilf-equivalent to τ. For example, if $\tau = 1232$, then $r(\tau) = 2321$. We call $\{\tau, r(\tau)\}$ the *symmetry class* of τ. Hence, to determine cardinality classes of patterns it is enough to consider only one representative from each symmetry class.

We also look at pattern avoidance on $\mathfrak{S}_{m_1 m_2 \dots m_k}$, the set of permutations of the multiset $S = 1^{m_1} 2^{m_2} \dots k^{m_k}$ with $m_i > 0$. Thus, $\mathfrak{S}_{m_1 m_2 \dots m_k}$ is the set of all words of length $m = m_1 + \cdots + m_k$ that contain the letter i exactly m_i times. For a given set of patterns T, we denote the set of permutations of the multiset S which avoid T by $\mathfrak{S}_{m_1 m_2 \dots m_k}(T)$.

3. Patterns of length 2

Even though the main focus of the paper is on patterns of length 3, we will state results for compositions and multisets restricted by patterns of length 2 for completeness. In this case, there are only two symmetry classes, with representatives 11 and 12. Avoiding 11 simply means having no repeated parts. Therefore, each parts occurs at most once, and the generating function is given by

$$C_{11}^{A}(x,y) = \prod_{a\in A}(1+x^a y).$$

A composition avoiding 12 is just a non-increasing string, so (see [4], Corollary 6.1)

$$C_{12}^{A}(x,y) = \frac{1}{\prod_{a\in A} 1-x^a y}.$$

For example, $C_{12}^{\mathbb{N}}(x) = \frac{1}{\prod_{j\geq 1} 1-x^j} = \sum_{n\geq 0} p_n x^n$, where p_n is the number of partitions of n.

For multisets, the results also follow easily since avoiding 11 means that each letter i has to occur exactly once (since, by definition, $m_i > 0$), and the k distinct letters can be arranged in $k!$ ways. On the other hand, avoiding 12 means that the letters have to be arranged in non-increasing order, i.e., in blocks of length m_i for each letter i. There is exactly one way to do so.

Theorem 3.1. *The number of permutations of the multiset $S = 1^{m_1}2^{m_2}\cdots k^{m_k}$ which avoid the patterns* 11 *and* 12*, respectively, are*

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(11)| = \begin{cases} k! & \text{if } m_i = 1 \quad \forall i \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad |\mathfrak{S}_{m_1 m_2 \ldots m_k}(12)| = 1.$$

4. Single patterns of length 3

For single patterns of length 3, there are eight symmetry classes for which we will use the following class representatives: 111, 112, 121, 221, 212, 123, 132, and 213. We will derive results for the first five symmetry classes, those where the pattern τ is not a permutation on [3]. The remaining three symmetry classes were considered by Savage and Wilf [6], and we will quote their results for completeness. Readers familiar with pattern avoidance for words will notice that there are more symmetry classes to be considered for compositions than for words. This stems from the fact that the complement map, $c(\sigma) = c(a_{i_1}a_{i_2}\ldots a_{i_m}) = a_{d+1-i_1}a_{d+1-i_2}\ldots a_{d+1-i_m}$ does not give Wilf equivalence for words, as the elements in the complement of σ do not sum to the same value as the elements of σ.

The pattern 111**.**

Theorem 4.1. *Let $A = \{a_1, \ldots, a_d\}$ be any ordered finite or infinite set of positive integers. Then*

$$\sum_{m\geq 0} C_{111}^{A}(x;m)\frac{y^m}{m!} = \prod_{a\in A}\left(1 + x^a y + \frac{1}{2}x^{2a}y^2\right).$$

Proof. Let σ be any composition in $C_{n;m}^{A}(111)$ and $A' = \{a_2, \ldots, a_d\}$. Since σ avoids 111, the number of occurrences of the letter a_1 in σ is 0, 1 or 2, and the number of such compositions is given by $C_{n;m}^{A'}(111)$, $m\,C_{n-a_1;m-1}^{A'}(111)$ and $\binom{m}{2}C_{n-2a_1;m-2}^{A'}(111)$, respectively. Hence, for all $n, m \geq 0$,

$$C_{n;m}^{A}(111) = C_{n;m}^{A'}(111) + mC_{n-a_1;m-1}^{A'}(111) + \binom{m}{2}C_{n-2a_1;m-2}^{A'}(111).$$

Multiplying by $\frac{1}{m!}x^n y^m$ and summing over all $n, m \geq 0$ we get that

$$\sum_{m\geq 0} C_{111}^{A}(x;m)\frac{y^m}{m!} = \left(1 + x^{a_1}y + \frac{1}{2}x^{2a_1}y^2\right)\sum_{m\geq 0} C_{111}^{A'}(x;m)\frac{y^m}{m!}.$$

Since $\sum_{m\geq 0} C_{111}^{\{a_1\}}(x;m)\frac{y^m}{m!} = 1 + x^{a_1}y + \frac{1}{2}x^{2a_1}y^2$, we get the desired result by induction on d. □

For multisets, we get a result that is similar to the case for the pattern 11. Avoiding 111 means that each letter i has to occur either once or twice (since, by definition, $m_i > 0$), and the $m_1 + \cdots + m_k$ letters can be arranged in $\frac{(m_1+\cdots+m_k)!}{m_1!\cdots m_k!}$ ways.

Theorem 4.2. *The number of permutations of the multiset* $S = 1^{m_1}2^{m_2}\cdots k^{m_k}$ *which avoid the pattern* 111 *is*

$$|\mathfrak{S}_{m_1 m_2 \dots m_k}(111)| = \begin{cases} \frac{(m_1+\cdots+m_k)!}{m_1!\cdots m_k!} & \text{if } m_i \leq 2 \quad \forall i \\ 0 & \text{otherwise.} \end{cases}$$

The patterns 112 and 121.

We start with the pattern 121. If a composition $\sigma \in C_{n;m}^{A}$ avoids the pattern 121, then it cannot contain parts other than a_1 between any two a_1's, which means that if σ contains two or more a_1's, then they have to be consecutive. Deletion of all a_1's from σ leaves another composition σ' which avoids 121 and contains no a_1's, so all a_2's in σ', if any, are consecutive. In general, deletion of all parts a_1 through a_j leaves a (possibly empty) composition $\tilde{\sigma}$ on parts a_{j+1} through a_d in which all parts a_{j+1}, if any, occur consecutively.

On the other hand, if a composition $\sigma \in C_{n;m}^{A}$ avoids the pattern 112, then only the leftmost a_1 of σ can occur before a greater part. The rest of the a_1's must occur at the end of σ. In fact, just as in the previous case, deletion of all parts a_1 through a_j leaves a (possibly empty) composition $\bar{\sigma}$ on parts a_{j+1} through a_d in which all occurrences of a_{j+1}, except possibly the leftmost one, are at the end of $\sigma^{(j)}$. We will call all occurrences of a part a_j, except the leftmost a_j, *excess* a_j's.

The preceding analysis suggests a natural bijection

$$\rho : C_{n;m}^{A}(121) \to C_{n;m}^{A}(112), \tag{4.1}$$

where ρ is defined by the following algorithm. Given a composition $\sigma \in C_{n;m}^{A}(121)$, we define $\sigma^{(0)} = \sigma$ and apply the following transformation of d steps. Let $\sigma^{(j-1)}$ be the composition that results from applying the transformation step $j-1$ times. Then define $\sigma^{(j)}$ to be the composition that results by cutting out the block of excess a_j's and inserting it immediately before the final block of all smaller excess parts in $\sigma^{(j-1)}$, or at the end of $\sigma^{(j-1)}$ if there are no smaller excess parts.

For example, for $\sigma = 22433111$, we have the following transformation:

$$22433111 \to 22433111 \to 24331211 \to 24313211 \to 24313211 = \rho(22433111).$$

It is easy to see that at the end of the algorithm, we get a composition $\sigma^{(d)} \in C_{n;m}^{A}(112)$.

The inverse map, $\rho^{-1} : C_{n;m}^{A}(112) \to C_{n,m}^{A}(121)$ is given by a similar algorithm of d steps. Given a composition $\sigma \in C_{n,m}^{A}(112)$ and keeping the same notation as above, the transformation step consists of cutting out the block of excess a_j's at the end of $\sigma^{(j-1)}$ and inserting it immediately after the leftmost a_j in $\sigma^{(j-1)}$.

Clearly, we get $\sigma^{(d)} \in C^A_{n;m}(121)$ at the end of the algorithm and have therefore shown that the patterns 121 and 112 are Wilf-equivalent on compositions.

We will now derive the generating function $C^A_{112}(x,y) = C^A_{121}(x,y)$. Consider all compositions $\sigma \in C^A_{n;m}(112)$ which contain at least one part a_1. The generating function for these compositions is given by

$$H^A_{112}(x,y) = C^A_{112}(x,y) - C^{A'}_{112}(x,y), \tag{4.2}$$

where $A' = \{a_2, \ldots, a_d\}$. On the other hand, each such σ either ends in a_1 or not. If σ ends in a_1, then deletion of this a_1 results in a composition in $\tilde{\sigma} \in C^A_{n-a_1;m-1}(112)$, since addition of a_1 at the right end to any composition in $\tilde{\sigma} \in C^A_{n-a_1;m-1}(112)$ does not produce an occurrence of the pattern 112.

If σ does not end in a_1, then it has no excess a_1's occurring at the right end of σ. Deletion of the single a_1 produces a composition $\tilde{\sigma} \in C^{A'}_{n-a_1;m-1}(112)$. Since insertion of a single a_1 into each such $\tilde{\sigma} \in C^{A'}_{n-a_1;m-1}(112)$ does not produce an occurrence of the pattern 112, we may insert a single a_1 in $m-1$ positions (all except the rightmost one) to get a composition $\sigma \in C^A_{n;m}(112)$ which contains a single a_1 that is not at the end. Thus, we have an alternative expression for$H^A_{112}(x,y)$:

$$\begin{aligned} H^A_{112}(x,y) &= x^{a_1} y\, C^A_{112}(x,y) + \sum_{n \geq a_1,\, m \geq 1} (m-1)|C^{A'}_{n-a_1;m-1}(112)| x^n y^m \\ &= x^{a_1} y\, C^A_{112}(x,y) + x^{a_1} y^2 \frac{\partial}{\partial y} C^{A'}_{112}(x,y). \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), we get the following theorem.

Theorem 4.3. *Let $A = \{a_1, \ldots, a_d\}$ be any ordered finite set of positive integers and let $A' = \{a_2, \ldots, a_d\}$. Then the patterns 112 and 121 are Wilf-equivalent on compositions, and*

$$(1 - x^{a_1} y) C^A_{112}(x,y) = C^{A'}_{112}(x,y) + x^{a_1} y^2 \frac{\partial}{\partial y} C^{A'}_{112}(x,y),$$

or, for all $m \geq 1$,

$$C^A_{112}(x;m) = C^{A'}_{112}(x;m) + x^{a_1} C^A_{112}(x;m-1) + (m-1) x^{a_1} C^{A'}_{112}(x;m-1),$$

where $C^A_{112}(x;0) = 1$ for any ordered set A.

For example, if $A = \{1, s\}$, then $A' = \{s\}$, $C^{A'}_{112}(x) = (1-x^s)^{-1}$ and Theorem 4.3 gives

$$C^A_{112}(x) = \frac{1 - x^s + x^{1+s}}{(1-x)(1-x^s)^2}.$$

In particular, if $A = \{1, 2\}$ then $C^A_{112}(x) = \frac{1-x+x^3}{(1-x)(1-x^2)^2}$, i.e., the number of 112 avoiding compositions is given by {1, 1, 2, 3, 4, 6, 7, 10, 11, 15, 16, 21, 22, 28, 29, 36, 37, 45, 46, 55, 56} for $n = 0 \ldots 20$. This sequence occurs in [7] as sequence A055802, where $|C^{\{1,2\}}_n(112)| = T(n+3, n-1)$. A closer look at this sequence shows remarkable structure, for which we will give a combinatorial explanation and an explicit formula for the odd and even terms.

For ease of notation we define $a(n) = |C^{\{1,2\}}_n(112)|$. The sequence above suggests that $a(2i) = a(2i-1) + 1$ and $a(2i+1) = a(2i) + i$. Since compositions avoiding 112 have only the leftmost 1 occurring before a larger part, they either have to end in a 1, or, if they end in a 2, they can have either no 1 or exactly one 1. Thus we can create the compositions of $n > 1$ avoiding 112 as follows:

(1) Append a 1 to each composition of $n-1$.
(2) Create the compositions of n ending in 2 as follows: If $n = 2i$, create the composition of all 2's (of which there is one). If $n = 2i+1$, then create the compositions of all 2's with a single 1 by inserting the single 1 at any of i positions (except the end), which creates an additional i compositions.

Considering the sequences for odd and even n separately, we obtain from the previous argument that for $i \geq 0$,

$$a(2i+1) = \sum_{j=1}^{i} j + (i+1) = (i+1)(i+2)/2 \qquad \text{and} \qquad a(2i) = \sum_{j=1}^{i-1} j + (i+1) = (i^2+i+2)/2.$$

The sequence of odd terms equals the triangular numbers (A000217 in [7]). Among the many combinatorial objects counted by this sequence are the permutations of $[n]$ which avoid 132 and have exactly one descent. The sequence of even terms did not previously occur in [7]. Using the formulas for odd and even n, we can verify that $a(n) = \frac{1}{16}(2n^2+6n+11-(2n-5)(-1)^n)$.

The reasoning that led to formulas for $a(n)$ for $A = \{1,2\}$ can be extended to $A = \{1,s\}$. If we let $b(n)$ denote the number of compositions on $A = \{1,s\}$ avoiding 112, then for $s = 4$, $\{b(n)\}_{n=0}^{20} = \{1,$ 1, 1, 1, 2, 3, 3, 3, 4, 6, 6, 6, 7, 10, 10, 10, 11, 15, 15, 15, 16$\}$, and we see that the values for $a(2i+1)$ are repeated three ($= s-1$) times. This is explained by extending the algorithm for creating the compositions recursively. The compositions have the same structure - they can end either in 1 or in s, and if they end in s, either one or no 1 occurs in the composition. Thus, we create the compositions of $n > 1$ with parts in $A = \{1,s\}$ avoiding 112 as follows:

(1) Append a 1 to each composition of $n-1$.
(2) Create the compositions of n ending in s as follows: If $n = s \cdot i$, create the composition of all s's (of which there is one). If $n = s \cdot i + 1$, then create the compositions of all s's with a single 1 by inserting the single 1 at any of i positions (except the end), which creates an additional i compositions.

If $n = s \cdot i + j$, where $2 \leq j \leq s-1$, then there are no compositions that end in s, as more than one 1 is necessary. Therefore,

$$b(s \cdot i) = a(2i) = (i^2+i+2)/2 \quad \text{and} \quad b(s \cdot i + j) = (i+1)(i+2)/2 \quad \text{for } j = 1, \ldots, s-1.$$

We now derive the corresponding results for the permutations of multisets avoiding patterns 112 and 121. The algorithm that was used to establish the Wilf-equivalence of 112 and 121 on compositions also applies to $\mathfrak{S}_{m_1 m_2 \ldots m_k}$, as the only action was a rearrangement of the parts of the composition. Therefore, $|\mathfrak{S}_{m_1 m_2 \ldots m_k}(112)| = |\mathfrak{S}_{m_1 m_2 \ldots m_k}(121)|$. To count the number of 112 avoiding permutations of the multiset S, we consider the permutation $\sigma = \sigma_1 \ldots \sigma_m$ ($m = m_1 + \cdots + m_k$) of $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ and focus on the positions of the 1's in σ. Since σ avoids the pattern 112, all 1's but the leftmost one have to occur as a block (of length $m_1 - 1$) at the end of σ. Let $\sigma' = \sigma_1 \ldots \sigma_{m+1-m_1}$; then σ avoids 112 if and only if σ' avoids 112. In addition, σ' contains only one letter 1 (the smallest letter in σ'), which can occur in $(m_2 + \cdots + m_k + 1)$ positions. Thus, the number of permutations $\sigma \in S$ avoiding 112 is given by $(m_2 + \cdots + m_k + 1)|\mathfrak{S}_{m_2 \ldots m_k}(112)|$, and we obtain the following result.

Theorem 4.4. *The number of permutations of the multiset $S = 1^{m_1}2^{m_2}\cdots k^{m_k}$ which avoid the patterns* 112 *and* 121*, respectively, is*

$$|\mathfrak{S}_{m_1m_2\ldots m_k}(112)| = |\mathfrak{S}_{m_1m_2\ldots m_k}(121)| = \prod_{j=2}^{k}(m_j + \cdots + m_k + 1).$$

The patterns 221 and 212.

We will now consider the patterns 221 and 212 and establish their Wilf-equivalence on compositions and on permutations of multisets. We obtain results that are very similar in structure to the results for the patterns 112 and 121.

Theorem 4.5. *Let $A = \{a_1, \ldots, a_d\}$ be any ordered finite set of positive integers and let $\tilde{A} = \{a_1, \ldots, a_{d-1}\}$. Then the patterns* 221 *and* 212 *are Wilf-equivalent on compositions, and*

$$(1 - x^{a_d}y)C^A_{221}(x, y) = C^{\tilde{A}}_{221}(x, y) + x^{a_d}y^2\frac{\partial}{\partial y}C^{\tilde{A}}_{221}(x, y).$$

Proof. Wilf equivalence can be proved in two different ways. First, we exhibit a bijection between $C^A_{n;m}(221)$ and $C^A_{n;m}(212)$ that relies on the bijection ρ^{-1} (see (4.1)) between $C^A_{n;m}(112)$ and $C^A_{n;m}(121)$ and on the notion of the *complement* of a composition. Let $\sigma = a_{i_1}a_{i_2}\ldots a_{i_m}$ be any composition in $C^A_{n;m}(221)$ such that the part a_i occurs in σ exactly r_i times. Then the *complement of σ with respect to A*, $c(\sigma) = c_A(\sigma)$ is defined by

$$c(\sigma) = a_{d+1-i_1}a_{d+1-i_2}\ldots a_{d+1-i_m}.$$

Clearly, $c(\sigma)$ is a 112-avoiding composition of n' with m parts in A such that the part a_i occurs exactly r_{d+1-i} times. Since c is one-to-one, $c^{-1}\circ\rho^{-1}\circ c$ is a bijection between the sets $C^A_{n;m}(221)$ and $C^A_{n;m}(212)$, showing Wilf-equivalence of the patterns 221 and 212.

The second way to establish a bijection between $C^A_{n;m}(221)$ and $C^A_{n;m}(212)$ is to adapt the arguments in the proof of Theorem 4.3 by replacing a_1 with a_d (and in general, a_j with a_{d-j+1}), "smallest" with "largest" and vice versa, and "112" with "221". This gives us the structure of the compositions avoiding 221 and 212 and a bijection ρ' between the sets $C^A_{n;m}(221)$ and $C^A_{n;m}(212)$, and the formula for $C^A_{221}(x, y)$ holds. □

For example, if $A = \{1, s\}$, then Theorem 4.5 gives $C^A_{221}(x) = \frac{1-x+x^{1+s}}{(1-x^s)(1-x)^2}$. For $s = 2$, the number of 221 avoiding compositions is given by {1, 1, 2, 3, 5, 7, 10, 13, 17, 21, 26, 31, 37, 43, 50, 57, 65, 73, 82, 91, 101} for $n = 0\ldots20$, which occurs in [7] as sequence A033638, and has closed form $|C^{\{1,2\}}_n(221)| = \frac{1}{8}(2n^2 + 7 + (-1)^n)$. This sequence also counts the number of (3412,123)-avoiding involutions in $\mathfrak{S}_n$ (see [3]) and exhibits a nice structure, namely the difference between consecutive pairs either remain the same or increase by one. This structure generalizes to sets $A = \{1, s\}$ in that blocks of s consecutive values all all have the same difference, and this difference increases by one from one block to the next. The only exception to this rule is the first block of values, which consists of $s-1$ values. For example, for $s = 4$, $\{|C^{\{1,4\}}_n(221)|\}^{20}_{n=0}$ = {1, 1, 1, 1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 19, 22, 25, 29, 33, 37, 41, 46}.

The combinatorial explanation and an explicit formula for $a(n) = |C^{\{1,s\}}_n(221)|$ are based on the structure of the 221 avoiding compositions. If a composition avoids 221, then any parts a_j that occur more than once have to be consecutive, except the leftmost one, which can occur before smaller parts.

The blocks of excess a_j's have to occur at the end, with the blocks in order of increasing parts from left to right. For $n < s$, there is exactly one composition consisting of all 1's. We create the compositions of $n \geq s$ avoiding 221 as follows:

(1) Prepend a 1 to each composition of $n-1$.
(2) Create the compositions of n beginning in s and containing the part s j times as follows: Start with s, followed by $n - j \cdot s$ 1's, then place the remaining $j-1$ s's.

A composition of n can have at most $\lfloor n/s \rfloor$ copies of s. Thus, $a(i \cdot s + \ell) - a((i \cdot s + \ell - 1) = i$ for $\ell = 0, \ldots, s-1$, and we derive that

$$a(n) = a(i \cdot s + \ell) = 1 + s\sum_{j=1}^{i-1} j + (\ell+1)i = \frac{2 + i(i-1)s + 2i(\ell+1)}{2} \quad \text{for } n \geq 0.$$

Note that the derivation is valid for $n \geq s$, but that the formula also holds for $0 \leq n < s$.

We now derive the corresponding results for the permutations of multisets avoiding the patterns 221 and 212, respectively. The algorithm that was used to establish the Wilf-equivalence of 221 and 212 on compositions also applies to $\mathfrak{S}_{m_1 m_2 \ldots m_k}$, as the only action was a rearrangement of the parts of the composition. Therefore, $|\mathfrak{S}_{m_1 m_2 \ldots m_k}(221)| = |\mathfrak{S}_{m_1 m_2 \ldots m_k}(212)|$. To count the number of 221 avoiding permutations of the multiset S, we consider the permutation $\sigma = \sigma_1 \ldots \sigma_m$ $(m = m_1 + \cdots + m_k)$ of $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ and focus on the positions of the k's in σ. Since σ avoids the pattern 221, all k's but the leftmost one have to occur as a block (of length $m_k - 1$) at the end of σ. Let $\sigma' = \sigma_1 \ldots \sigma_{m+1-m_k}$; then σ avoids 221 if and only if σ' avoids 221. In addition, σ' contains only one letter k (the maximal letter in σ'), which can occur in $(m_1 + \cdots + m_{k-1} + 1)$ positions. Thus, the number of permutations $\sigma \in S$ avoiding 221 is given by $(m_1 + \cdots + m_{k-1} + 1)|\mathfrak{S}_{m_1 m_2 \ldots m_{k-1}}(221)|$, and we obtain the following result.

Theorem 4.6. *The number of permutations of the multiset $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ which avoid the patterns 221 and 212, respectively, is*

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(221)| = |\mathfrak{S}_{m_1 m_2 \ldots m_k}(212)| = \prod_{j=1}^{k-1} (m_1 + \cdots + m_j + 1).$$

The patterns 123, 132, and 213

As noted earlier, results on these patterns for $A = \mathbb{N}$ were obtained by Savage and Wilf, and are stated here for reference.

Theorem 4.7 (Theorems 1 and 3 [6])**.** *For $A = \mathbb{N}$, the patterns 123, 132, and 213 are Wilf-equivalent on compositions and on multisets. The generating function for the number of compositions avoiding a pattern $\tau \in \mathfrak{S}_3$ is given by*

$$C_\tau^{\mathbb{N}}(x) = \sum_{i \geq 1} \frac{1}{1-x^i} \prod_{j \neq i} \left\{ \frac{1-x^i}{(1-x^{j-i})(1-x^i-x^j)} \right\}.$$

5. Pairs of patterns of length three

We now classify sets of two patterns of length three on $\{1, 2\}$ and determine their equivalence classes and generating functions. Note that if τ_1 and τ_2 are two patterns, then any composition that avoids $\{\tau_1, \tau_2\}$ will also avoid $\{r(\tau_1), r(\tau_2)\}$, where r is the reversal map defined in Section 2. Using this argument, the 21 possible pairs formed from the seven patterns of length 3 on $\{1,2\}$ are reduced to 13 symmetry classes:

$$\{111,112\},\{111,121\},\{111,212\},\{111,221\},\{112,121\},\{112,122\},\{112,211\},$$
$$\{112,212\},\{112,221\},\{121,212\},\{122,212\},\{122,221\},\{122,121\}.$$

We will show that the first and second pair of this list are Wilf-equivalent, and likewise, the third and fourth pair. No other pairs are in the same equivalence class.

The patterns $\{111,112\},\{111,121\},\{111,212\}$**, and** $\{111,221\}$**.**

Theorem 5.1. *Let $A=\{a_1,a_2,\ldots,a_d\}$ be any ordered finite set of positive integers. Then the pairs $\{111,112\}$ and $\{111,121\}$ are Wilf-equivalent on compositions, and for all $m\geq 0$,*

$$C^{A}_{111,112}(x;m)=C^{A'}_{111,112}(x;m)+m\,x^{a_1}C^{A'}_{111,112}(x;m-1)+(m-1)x^{2a_1}C^{A'}_{111,112}(x;m-2),$$

where $A'=\{a_2,\ldots,a_d\}$.

Proof. To prove equivalence, note that the bijection $\rho: C^A_{n;m}(121)\to C^A_{n;m}(112)$ of (4.1) preserves the number of excess copies of each part and that avoiding the pattern 111 is the same as having at most one excess part a_j for each $j=1,\ldots,d$. Thus, the restriction of ρ to compositions with at most one excess part of each kind yields a bijection $\rho_{|111}: C^A_{n;m}(111,121)\to C^A_{n;m}(111,112)$.

To derive the generating function we consider compositions $\sigma\in C^A_{n;m}(111,112)$ that contain i copies of a_1. Since σ avoids 111, $i\in\{0,1,2\}$. If there are two copies of a_1, then one of them has to be at the end because σ avoids 112, and the other a_1 can occur at any of $m-1$ positions. Corresponding to the three cases, the generating functions for the number of such compositions σ are $C^{A'}_{111,112}(x;m)$, $mx^{a_1}C^{A'}_{111,112}(x;m-1)$ and $(m-1)x^{2a_1}C^{A'}_{111,112}(x;m-2)$, respectively, which gives the stated result for $C^A_{111,112}(x;m)$. □

Replacing a_1 by a_d in the arguments in the proof of Theorem 5.1, we obtain a bijection between the sets $C^A_{n;m}(111,221)$ and $C^A_{n;m}(111,212)$, and we obtain the following result.

Theorem 5.2. *Let $A=\{a_1,a_2,\ldots,a_d\}$ be any ordered finite set of positive integers. Then the pairs $\{111,212\}$ and $\{111,221\}$ are Wilf-equivalent on compositions, and for all $m\geq 0$,*

$$C^{A}_{111,212}(x;m)=C^{\tilde{A}}_{111,212}(x;m)+mx^{a_d}C^{\tilde{A}}_{111,212}(x;m-1)+(m-1)x^{2a_d}C^{\tilde{A}}_{111,212}(x;m-2),$$

where $\tilde{A}=\{a_1,\ldots,a_{d-1}\}$.

We now derive the results for multisets.

Theorem 5.3. *The number of permutations of the multiset $S=1^{m_1}2^{m_2}\cdots k^{m_k}$ which avoid the patterns $\{111,112\}$, $\{111,121\}$, $\{111,212\}$, and $\{111,221\}$ are*

$$\begin{aligned}|\mathfrak{S}_{m_1m_2\ldots m_k}(111,112)| &= |\mathfrak{S}_{m_1,m_2\ldots m_k}(111,121)|\\ &= \begin{cases} 0 & \text{if there exists } i \text{ such that } m_i\geq 3\\ \prod_{j=2}^{k}(m_j+\cdots+m_k+1) & \text{otherwise.}\end{cases}\end{aligned}$$

and

$$|\mathfrak{S}_{m_1m_2\ldots m_k}(111,212)| = |\mathfrak{S}_{m_1,m_2\ldots m_k}(111,221)| = \begin{cases} 0 & \text{if there exists } i \text{ such that } m_i \geq 3 \\ \prod_{j=1}^{k-1}(m_1+\cdots+m_j+1) & \text{otherwise.} \end{cases}$$

Proof. As in the case of the respective (non-111) single patterns, the bijection $\rho_{|_{111}} : C^A_{n;m}(111,121) \to C^A_{n;m}(111,112)$ (and the corresponding bijection between $C^A_{n;m}(111,212)$ and $C^A_{n;m}(111,221)$) also applies to multisets and proves Wilf-equivalence. Avoiding 111 restricts the multisets to those where $m_i \leq 2$ for all i. On these multisets, avoiding 112 (221, respectively) is the only restriction, and Theorems 4.4 and 4.6 give the stated results. □

We will now look at the remaining patterns. Even though none of these are Wilf-equivalent, they are grouped together in pairs because of the similarity in the structure of their respective generating functions.

The patterns $\{112,121\}$ and $\{122,212\}$.

Theorem 5.4. *Let $A = \{a_1, a_2, \ldots, a_d\}$ be any ordered finite set of positive integers. Then for any $m \geq 0$,*

$$C^A_{112,121}(x;m) = C^{A'}_{112,121}(x;m) + m\,x^{a_1} C^{A'}_{112,121}(x;m-1) + \sum_{j=2}^{m} x^{ja_1} C^{A'}_{112,121}(x;m-j)$$

and

$$C^A_{122,212}(x;m) = C^{\tilde{A}}_{122,212}(x;m) + m\,x^{a_d} C^{\tilde{A}}_{122,212}(x;m-1) + \sum_{j=2}^{m} x^{ja_d} C^{\tilde{A}}_{122,212}(x;m-j),$$

where $A' = \{a_2, \ldots, a_d\}$ and $\tilde{A} = \{a_1, \ldots, a_{d-1}\}$.

Proof. Let $\sigma \in C^A_{n;m}(112,121)$ be a composition that contains j parts a_1. Since σ avoids 112 and 121, we have that for $j > 1$, all j copies of part a_1 appear in σ in positions $m-j+1$ through m. When $j = 1$, the single a_1 may appear in any position. Therefore,

$$C^A_{112,121}(x;m) = C^{A'}_{112,121}(x;m) + m\,x^{a_1} C^{A'}_{112,121}(x;m-1) + \sum_{j=2}^{m} x^{ja_1} C^{A'}_{112,121}(x;m-j),$$

as claimed. Replacing a_1 with a_d in the above arguments gives the result for $C^A_{122,212}(x;m)$. □

The results for multisets are given by the following theorem.

Theorem 5.5. *The number of permutation of the multiset $S = 1^{m_1}2^{m_2}\cdots k^{m_k}$ which avoid $\{112,121\}$ and $\{122,212\}$, respectively, are*

$$|\mathfrak{S}_{m_1m_2\ldots m_k}(112,121)| = \prod_{j=1}^{k-1} b_j, \quad \text{where } b_j = \begin{cases} (m_j+\cdots+m_k) & \text{if } m_j = 1 \\ 1 & \text{otherwise} \end{cases}$$

and

$$|\mathfrak{S}_{m_1m_2\ldots m_k}(122,212)| = \prod_{j=2}^{k} c_j, \quad \text{where } c_j = \begin{cases} (m_1+\cdots+m_j) & \text{if } m_j = 1 \\ 1 & \text{otherwise} \end{cases}.$$

Proof. Let us consider the case $\mathfrak{S}_{m_1 m_2 \dots m_k}(112,121)$. As in the proof of Theorem 5.4, if $m_1 = 1$, then the single 1 can occur in any of the $m_2 + \cdots + m_k + 1 = m_1 + \cdots + m_k$ positions, and we get that $|\mathfrak{S}_{1 m_1 m_2 \dots m_k}(112,121)| = (m_1 + m_2 + \cdots + m_k)|\mathfrak{S}_{m_2 \dots m_k}(112,121)|$. If $m_1 \geq 2$, then all the 1's have to occur at the right end, and therefore $|\mathfrak{S}_{m_1 m_2 \dots m_k}(112,121)| = |\mathfrak{S}_{m_2 \dots m_k}(112,121)|$. Using that $|\mathfrak{S}_{m_k}(112,121)| = 1$, we obtain the desired result by induction on k. A similar argument works for the permutations on S avoiding $\{122,212\}$, except that we now start the argument with the largest part k. □

The patterns $\{112,211\}$ and $\{122,221\}$.

Theorem 5.6. *Let $A = \{a_1, a_2, \dots, a_d\}$ be any ordered finite set of positive integers. Then for all $m \geq 0$,*

$$C^A_{112,211}(x;m) = C^{A'}_{112,211}(x;m) + m\,x^{a_1} C^{A'}_{112,211}(x;m-1) + x^{2a_1} C^{A'}_{112,211}(x;m-2) + x^{m\,a_1}$$

and

$$C^A_{122,221}(x;m) = C^{\tilde{A}}_{122,221}(x;m) + m\,x^{a_d} C^{\tilde{A}}_{122,221}(x;m-1) + x^{2a_d} C^{\tilde{A}}_{122,221}(x;m-2) + x^{m\,a_d},$$

where $A' = \{a_2, \dots, a_d\}$ and $\tilde{A} = \{a_1, \dots, a_{d-1}\}$.

Proof. Let $\sigma \in C^A_{n;m}(112,211)$ be a composition that contains j parts a_1. Since σ avoids 112 and 211, $j = 0,1,2,m$. When $j = 2$, the two a_1's must be at the beginning and at the end. Hence, it is easy to see that for $j = 0,1,2,m$ the generating function for such compositions are $C^{A'}_{112,211}(x;m)$, $m\,x^{a_1} C^{A'}_{112,211}(x;m-1)$, $x^{2a_1} C^{A'}_{112,211}(x;m-2)$ and $x^{m\,a_1}$, respectively. Therefore, for all $m \geq 0$,

$$C^A_{112,211}(x;m) = C^{A'}_{112,211}(x;m) + m x^{a_1} C^{A'}_{112,211}(x;m-1) + x^{2a_1} C^{A'}_{112,211}(x;m-2) + x^{m a_1},$$

as claimed. Replacing a_1 with a_d in the above argument gives the result for $C^A_{122,221}(x;m)$. □

Again, the results for multisets follow easily along the lines of the proof for compositions.

Theorem 5.7. *The number of permutation of the multiset $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ which avoid $\{112,211\}$ and $\{122,221\}$, respectively, are*

$$|\mathfrak{S}_{m_1 m_2 \dots m_k}(112,211)| = \prod_{j=1}^{k-1} b_j, \quad \textit{where } b_j = \begin{cases} (m_j + \cdots + m_k) & \textit{if } m_j = 1 \\ 1 & \textit{if } m_j = 2 \\ 0 & \textit{if } m_j \geq 2 \end{cases}$$

and

$$|\mathfrak{S}_{m_1 m_2 \dots m_k}(122,221)| = \prod_{j=2}^{k} c_j, \quad \textit{where } c_j = \begin{cases} (m_1 + \cdots + m_j) & \textit{if } m_j = 1 \\ 1 & \textit{if } m_j = 2 \\ 0 & \textit{if } m_j \geq 2 \end{cases}.$$

Proof. As stated in the proof of Theorem 5.5, if the permutation avoids $\{112,211\}$, then $m_1 \in \{1,2\}$. The single 1 can occur in any of the $m_1 + \cdots + m_k$ positions and two 1's have to occur at the beginning and the end. Thus, $|\mathfrak{S}_{m_1 m_2 \dots m_k}(112,121)| = b_1 |\mathfrak{S}_{m_2 \dots m_k}(112,121)|$. Induction on k proves the desired result since $|\mathfrak{S}_m| = 1$ for any m. A similar argument works for the permutations on S avoiding $\{122,221\}$, except that the argument starts with the largest part k. □

The patterns $\{112, 212\}$ and $\{122, 121\}$.

Theorem 5.8. *Let $A = \{a_1, a_2, \ldots, a_d\}$ be any ordered finite set of positive integers. Then the generating function $C^A_{112,212}(x, y)$ is given by*

$$\prod_{j=1}^{d}\frac{1+x^{a_j}y}{1-x^{a_j}y}-\sum_{i=1}^{d}\left[\frac{x^{a_i}y}{1-x^{a_i}y}\prod_{j=1}^{i-1}\frac{1+x^{a_j}y}{1-x^{a_j}y}\left(1-\sum_{U\sqcup V=A'_i}(C^U_{112,212}(x,y)-1)(C^V_{112,212}(x,y)-1)\right)\right]$$

and the generating function $C^A_{122,121}(x, y)$ is given by

$$\prod_{j=1}^{d}\frac{1+x^{a_j}y}{1-x^{a_j}y}-\sum_{i=1}^{d}\left[\frac{x^{a_i}y}{1-x^{a_i}y}\prod_{j=i+1}^{d}\frac{1+x^{a_j}y}{1-x^{a_j}y}\left(1-\sum_{U\sqcup V=\tilde{A}_i}(C^U_{122,121}(x,y)-1)(C^V_{122,121}(x,y)-1)\right)\right],$$

where $A'_i = \{a_{i+1}, \ldots, a_d\}$, $\tilde{A}_i = \{a_1, \ldots, a_{i-1}\}$, $C^{\varnothing}_T(x, y) = 1$, and $U \sqcup V = D$ denotes the collection of sets U and V such that $U \cup V = D$ and $U \cap V = \emptyset$.

Proof. Let $A' = A'_1 = \{a_2, \ldots, a_d\}$ and $\sigma \in C^A_{n;m}(112, 212)$ have exactly j parts a_1. If $j = 0$, the generating function for such compositions is $C^{A'}_{112,212}(x, y)$. If $j \geq 1$, then we focus on the positions where the part a_1 occurs. Since σ avoids 112, all but the leftmost a_1 have to occur as a block at the right end, thus, $\sigma = \sigma^1 a_1 \sigma^2 a_1 \ldots a_1$, where σ^i and the block of a_1's may be empty. Furthermore, since σ avoids 212, the parts in σ^1 have to be distinct from the parts that occur in σ^2. Therefore, σ (with parts in A) avoids $\{112, 212\}$ if and only if σ^1 and σ^2 are $\{112, 212\}$-avoiding on sets U and V, respectively, such that $U \sqcup V = A'$. To compute the generating function for $j \geq 1$, we have to consider three cases: $\sigma^1 = \sigma^2 = \emptyset$, exactly one of σ^1 and σ^2 is the empty set, and neither σ^1 nor σ^2 is empty. Therefore, it is easy to see that

$$C^A_{112,212}(x,y)=C^{A'}_{112,212}(x,y)+\frac{x^{a_1}y}{1-x^{a_1}y}\left[1+2\left(C^{A'}_{112,212}(x,y)-1\right)+\sum_{U\sqcup V=A'}(C^U_{112,212}(x,y)-1)(C^V_{112,212}(x,y)-1)\right].$$

Collecting terms and simplifying gives that $C^A_{112,212}(x, y)$ is given by

$$\frac{1+x^{a_1}y}{1-x^{a_1}y}C^{A'}_{112,212}(x,y)-\frac{x^{a_1}y}{1-x^{a_1}y}\left[1-\sum_{U\sqcup V=A'}(C^U_{112,212}(x,y)-1)(C^V_{112,212}(x,y)-1)\right].$$

Since $C^{\{a\}}_{112,212}(x, y) = 1/(1 - x^a y)$, the result follows by induction on d. Replacing a_1 with a_d and A'_i with $\tilde{A}_i$ in the above proof gives the result for $C^A_{122,121}(x, y)$. □

As an example, we consider the set $A = \{1, s\}$. Then Theorem 5.8 gives

$$C^A_{112,212}(x)=\frac{1+x}{1-x}\cdot\frac{1+x^s}{1-x^s}-\frac{x}{1-x}\cdot\frac{1+x^s}{1-x^s}-\frac{x^s}{1-x^s}=\frac{1+x^{s+1}}{(1-x)(1-x^s)}.$$

For $s = 2$, the sequence for the number of compositions avoiding $\{112, 212\}$ is given by {1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20} for $n = 0, \ldots, 20$, and for $s = 4$, the corresponding sequence is given by {1, 1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 7, 7, 8, 9, 9, 9, 10}. The pattern that is suggested by these two examples is that the number of compositions increases by 1 if $n = k \cdot s$ and $n = k \cdot s + 1$ (for $k \geq 1$), and that the number of compositions equals 1 for $n < s$. This pattern can be explained combinatorially as follows. Since all 1's but the leftmost one have to occur in a block at the end of the composition, and the parts to the left and right of that leftmost 1 have to be different, then the compositions either end in 1, or if they end in s, then they have to consist of all s's or of a 1 followed by all s's. Thus we create the compositions recursively as follows:

(1) Append a 1 to the end of the compositions of $n-1$.
(2) If $n = k\cdot s$, create the composition $ss\dots ss$. If $n = k\cdot s+1$, create the composition $1ss\dots ss$.

This algorithm gives an explicit formula for the number of compositions avoiding 112 and 212 simultaneously:

$$|C_n^{\{1,s\}}(112,212)| = \begin{cases} 1 & \text{for } n = 0,\dots,s-1; \\ 2k & \text{for } n = k\cdot s, k\geq 1; \\ 2k+1 & \text{for } n = k\cdot s+1,\dots,(k+1)\cdot s-1, k\geq 1. \end{cases}$$

We now state the results for multisets which follow using the arguments in the proof of Theorem 5.8.

Theorem 5.9. *The number of permutation of the multiset $S = 1^{m_1}2^{m_2}\cdots k^{m_k}$ which avoid $\{112,212\}$ and $\{122,121\}$, respectively, are*

$$|\mathfrak{S}_{m_1m_2\dots m_k}(112,212)| = \sum_{Q\sqcup P=\{2,\dots,k\}} |\mathfrak{S}_{m_{q_1}\dots m_{q_s}}(112,212)||\mathfrak{S}_{m_{p_1}\dots m_{p_{k-s-1}}}(112,212)|$$

and

$$|\mathfrak{S}_{m_1m_2\dots m_k}(122,121)| = \sum_{Q\sqcup P=\{1,\dots,k-1\}} |\mathfrak{S}_{m_{q_1}\dots m_{q_s}}(122,121)||\mathfrak{S}_{m_{p_1}\dots m_{p_{k-s-1}}}(122,121)|,$$

where $Q = \{q_1,\dots,q_s\}$, $P = \{p_1,\dots,p_{k-1-s}\}$, and the number of permutations of the empty multiset are defined to be 1.

The patterns $\{121,212\}$ and $\{112,221\}$.

Even though the structure of the generating functions looks similar, the arguments in the proof for $\{112,221\}$ go beyond replacing a_1 with a_d in the proof for $\{121,212\}$ because these pairs of patterns are not Wilf-equivalent for words, unlike the other pairs that were grouped together in this section.

Theorem 5.10. *Let A be any ordered set of positive integers. Then*

$$C^A_{121,212}(x,y) = 1 + \sum_{\varnothing\neq B\subset A}\left(|B|!\prod_{b\in B}\frac{x^b y}{1-x^b y}\right),$$

and the number of permutations of the multiset $S = 1^{m_1}2^{m_2}\cdots k^{m_k}$ which avoid $\{121,212\}$ is given by

$$|\mathfrak{S}_{m_1m_2\dots m_k}(121,212)| = k!.$$

Proof. Let $\sigma \in C^A_{n;m}(121,212)$ contain exactly j distinct parts. Then all copies of each part a_{i_1} through a_{i_j} must be consecutive, or σ would contain an occurrence of either 121 or 212. Hence, σ is a concatenation of j constant strings. Suppose the i-th string has length $m_i > 0$, then $m = \sum_{i=1}^{j} m_i$. Therefore, to obtain any $\sigma \in C^A_{n;m}(121,212)$, we can choose a subset B of A with j parts such that the composition σ contains j strings of the form $bb\dots b$ where $b\in B$. Each such selection of a subset B results in $j! = |B|!$ compositions, which yields the desired formula for the generating function. The result for multisets follows from the arguments above since there are k distinct parts in each of the permutations. □

Theorem 5.11. *Let $A = \{a_1, a_2, \ldots, a_d\}$ be any ordered finite set of positive integers. Then*

$$C^A_{112,221}(x,y) = 1 + \sum_{\varnothing \neq B \subset A} \left(|B|! \prod_{b \in B} (x^b y) \sum_{b' \in B} \frac{1}{1 - x^{b'} y} \right),$$

and the number of permutations of the multiset $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ which avoid $\{112, 221\}$ is given by

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(112, 221)| = \begin{cases} 0, & \text{if there exist i,j such that } m_i, m_j \geq 2; \\ k! & \text{otherwise.} \end{cases}$$

Proof. Let $\sigma \in C^A_{n;m}(112, 221)$ and let $j \leq m$ be the largest index such that $\sigma_1, \sigma_2, \ldots, \sigma_j$ are all distinct. If $j < m$, then σ_{j+1} repeats one of the preceding parts, and the parts to the right of σ_{j+1}, if any, have to be equal to σ_{j+1} since σ avoids 112 and 221. Therefore, the compositions avoiding $\{112, 221\}$ can be created as follows: Select a nonempty set $B \subset A$. Then the elements in B can be arranged in $|B|!$ ways, and each such arrangement contributes the term $\prod_{b \in B} x^b y$ to the generating function. The composition can be extended with $j \geq 0$ copies of a specific $b' \in B$, which accounts for the inner sum. The result for multisets follows easily from the preceding discussion since avoiding $\{112, 221\}$ means that only one of the parts 1 through k can occur more than once. □

The pattern $\{112, 122\}$.

Unfortunately, the case of the pair (112,122) remains unsolved.

6. Some patterns of arbitrary length

We consider two types of patterns of arbitrary length. The first one is the generalization of the pattern 111, and the second one generalizes 121. We denote the pattern consisting of ℓ 1's by $\langle 1 \rangle_\ell$, and the pattern with s 1's (respectively, t 1's) to the left (respectively, right) of the single 2 by $v_{s,t}$.

The pattern $11\ldots1$

The results for the pattern $11\ldots1$ are straightforward generalizations of Theorems 4.1 and 4.2. Avoiding $\langle 1 \rangle_\ell$ means that each part a_j can occur at most $\ell - 1$ times in the composition, and if a part occurs j times among the m parts of the composition, then there are $\binom{m}{j}$ possible compositions. For multisets, avoidance of $\langle 1 \rangle_\ell$ means that each part can occur at most $\ell - 1$ times.

Theorem 6.1. *For any $\ell \geq 1$ and any finite ordered set of positive integers A,*

$$\sum_{m \geq 0} C^A_{\langle 1 \rangle_\ell}(x;m) \frac{y^m}{m!} = \prod_{a \in A} \left(\sum_{j=0}^{\ell-1} \frac{x^{ja} y^j}{j!} \right),$$

and the number of permutations of the multiset $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ which avoid $\langle 1 \rangle_\ell$ is given by

$$|\mathfrak{S}_{m_1 m_2 \ldots m_k}(\langle 1 \rangle_\ell)| = \begin{cases} \frac{(m_1 + \cdots + m_k)!}{m_1! \cdots m_k!} & \text{if } m_i \leq \ell - 1 \quad \forall i; \\ 0 & \text{otherwise.} \end{cases}$$

The pattern $11\ldots121\ldots11$

We derive the generating function for the number of compositions avoiding $v_{s,t}$ and show that all patterns $v_{s,t}$ of the same length $(s + t + 1)$ are Wilf-equivalent.

Theorem 6.2. *Let $A = \{a_1, \ldots, a_d\}$ be any finite ordered set of positive integers. Then for all $m \geq 1$,*

$$C^A_{v_{s,t}}(x; m+1) - x^{a_1} C^A_{v_{s,t}}(x; m) = \sum_{j=0}^{s+t-1} x^{j a_1} \binom{m}{j} C^{A'}_{v_{s,t}}(x; m+1-j).$$

Proof. Let $\sigma \in C^A_{n;m}(v_{s,t})$ contain exactly j parts a_1. If $j \leq s+t-1$, then the a_1's cannot be part of an occurrence of $v_{s,t}$ in σ. Therefore, these a_1's can be in any j positions, so the generating function for number of such compositions is $x^{j\,a_1}\binom{m}{j} C^{A'}_{v_{s,t}}(x; m-j)$, where $A' = \{a_2, \ldots, a_d\}$. If $j \geq s+t$, then the s-th through $(j-t+1)$-st ($= t$-th from the right) a_1's must be consecutive parts in σ. (If $s = 0$, then the block of $j-t+1$ a_1's must occur on the left end, and if $t = 0$, then the s-th through the j-th a_1's must occur at the right end.) Hence, the generating function for number of such compositions is $x^{j\,a_1}\binom{m-j+s+t-1}{s+t-1} C^{A'}_{v_{s,t}}(x; m-j)$, thus, for all $m \geq 1$,

$$C^A_{v_{s,t}}(x; m) = \sum_{j=0}^{s+t-1} x^{j a_1} \binom{m}{j} C^{A'}_{v_{s,t}}(x; m-j) + \sum_{j=s+t}^{m} x^{j a_1} \binom{m-j+s+t-1}{s+t-1} C^{A'}_{v_{s,t}}(x; m-j), \tag{6.1}$$

or, equivalently,

$$C^A_{v_{s,t}}(x; m+1) - x^{a_1} C^A_{v_{s,t}}(x; m) = \sum_{j=0}^{s+t-1} x^{j a_1} \binom{m}{j} C^{A'}_{v_{s,t}}(x; m+1-j),$$

where the last equation follows from substituting (6.1) into the left-hand side, re-indexing the sums that occur in $x^{a_1} C^A_{v_{s,t}}(x; m)$, combining like terms and simplifying. □

Since the expression for $C^A_{v_{s,t}}(x; m)$ depends only on $s+t$, we obtain Wilf-equivalence as an immediate corollary.

Corollary 6.3. *Let $s + t \geq 1$. Then for any ordered finite set A of positive integers and for all $m, n \geq 0$*

$$|C^A_{n;m}(v_{s,t})| = |C^A_{n;m}(v_{s+t,0})|.$$

In other words, all patterns $v_{s,t}$ with the same $s+t$ are Wilf-equivalent.

Note that for $s+t = 2$ we obtain that the patterns 112, 121, and 211 are Wilf-equivalent. 112 and 211 are in the same symmetry class, and the Wilf-equivalence of 112 and 121 was proved in Theorem 4.3 using the bijection ρ defined in (4.1). We can give an alternative proof of the corollary by generalizing the bijection ρ: Let $\sigma^{(0)} = \sigma \in C^A_{n;m}(v_{s,t})$ with $t \neq 0$ and let $\sigma^{(i)}$ be the composition that results after applying Step i to $\sigma^{(i-1)}$. Suppose that $\sigma^{(i-1)}$ contains j parts a_i.

Step i:

If $j \leq s+t-1$, do nothing. If $j \geq s+t$, then remove the $(s+1)$-st through $(j-t+1)$-st a_i (the *excess a_i's* - all but the leftmost a_i in the block) and insert them immediately before the final block of all smaller excess parts, or at the end if there are no smaller excess parts .

Since there are only $(s-1)+1+(t-1) = s+t-1$ a_i's that can occur before a larger part, $\sigma^{(i)}$ avoids $v_{s+t,0}$. Clearly, this map is reversible, and $\sigma^{|A|} \in C^A_{n;m}(v_{s+t,0})$.

For example, let $\sigma = 413211233$, which avoids 1121, i.e., $s = 2, t = 1$. Then application of the steps described above yields this sequence of transformations:

$$413211332 \to 413213321 \to 413213321 \to 4132123231 \to 4132123231.$$

We now derive the corresponding results for the permutations of multisets avoiding the pattern $v_{s,t}$. The algorithm that was used to establish the Wilf-equivalence of $v_{s,t}$ and $v_{s+t,0}$ on compositions also applies to $\mathfrak{S}_{m_1 m_2 \dots m_k}$, as the only action was a rearrangement of the parts of the composition. Therefore, $|\mathfrak{S}_{m_1 m_2 \dots m_k}(v_{s,t})| = |\mathfrak{S}_{m_1 m_2 \dots m_k}(v_{s+t,0})|$. To count the number of $v_{s+t,0}$ avoiding permutations of the multiset S, we consider the permutation $\sigma = \sigma_1 \dots \sigma_m$ $(m = m_1 + \cdots + m_k)$ of $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ and focus on the positions of the 1's in σ. Since σ avoids the pattern $v_{s+t,0}$, then we have to distinguish between the two cases $m_1 \leq s+t-1$ and $m_1 \geq s+t$. If $m_1 \leq s+t-1$, then the m_1 1's can occur in any of the m positions and the number of permutations $\sigma \in S$ avoiding $v_{s+t,0}$ is given by $\binom{m_1+\cdots+m_k}{m_1} |\mathfrak{S}_{m_2 \dots m_k}(v_{s+t,0})|$. If $m_1 \geq s+t$, then $m_1 - (s+t-1)$ 1's have to occur as a block at the end of σ. Let $\sigma' = \sigma_1 \dots \sigma_{m-m_1-s-t+1}$; then σ avoids $v_{s+t,0}$ if and only if σ' avoids $v_{s+t,0}$. In addition, σ' contains only $s+t-1$ 1's (the smallest letter in σ'), which can occur in $m_2 + \cdots + m_k + s + t - 1$ positions and the number of permutations $\sigma \in S$ avoiding $v_{s+t,0}$ is given by $\binom{m_2+\cdots+m_k+s+t-1}{s+t-1} |\mathfrak{S}_{m_2 \dots m_k}(v_{s+t,0})|$. Therefore, we obtain the following result.

Theorem 6.4. *The number of permutations of the multiset $S = 1^{m_1} 2^{m_2} \cdots k^{m_k}$ which avoid $v_{s+t,0}$ is given by*

$$|\mathfrak{S}_{m_1 m_2 \dots m_k}(v_{s+t,0})| = |\mathfrak{S}_{m_1 m_2 \dots m_k}(v_{s,t})| = \prod_{j=1}^{k-1} \binom{m_{j+1} + \ldots + m_k + \min\{m_j, s+t-1\}}{\min\{m_j, s+t-1\}}.$$

7. Summary

We have provided a complete analysis for the number of compositions and permutations on multisets restricted by a single pattern or by a pair of patterns of length 3 on the alphabet $\{1, 2\}$, with the exception of the pattern $\{112, 122\}$. Furthermore, we have given results for generalizations to single patterns of arbitrary length consisting of all 1's or those containing a single 2.

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