Nim, Wythoff and beyond - let's play!

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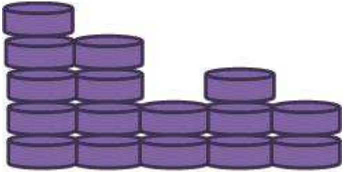
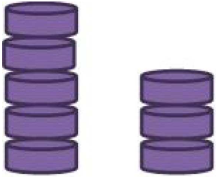
Nim and Wythoff

- **Nim**: Select one of the n stacks, take at least one token



- **Wythoff**: Take any number of tokens from **one** stack OR select the **same** number of tokens from both stacks



How to win????

Question: For a given starting position (= heights of the stacks) in a game, can we determine whether Player I or Player II has a winning strategy, that is, can make moves in such a way that s/he will win, no matter how the other player plays? (Last player to move wins)

Goal: Determine the **set of losing positions**, that is, all positions that result in a loss for the player playing from that position.

Smaller Goal: Say something about the **structure** of the losing positions.

Combinatorial Games

Definition

An *impartial combinatorial game* has the following properties:

- each player has the **same moves** available at each point in the game (as opposed to chess, where there are white and black pieces).
- **no randomness** (dice, spinners) is involved and each player has **complete information** about the game and the potential moves

Analyzing Nim and Wythoff

Definition

A *position* in the game is denoted by $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, where $p_i \geq 0$ denotes the number of tokens in stack i. A position that can be reached from the current position by a legal move is called an *option*. The directed graph which has the positions as the nodes and an arrow between a position and its options is called the *game graph*.

We do not distinguish between a position and any of its rearrangements. We will use the position that is ordered in decreasing order as the representative.

Options of position $(3, 2)$ in Nim and Wythoff

$(3, 2) \quad \rightsquigarrow$

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Additional moves for Wythoff

$(3, 2) \quad \rightsquigarrow$

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Overall

$(3, 2) \quad \rightsquigarrow \quad (3, 1), (3, 0), (2, 2), (2, 1), (2, 0)$ for Nim

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Game graph for position $(3, 2)$ for Nim

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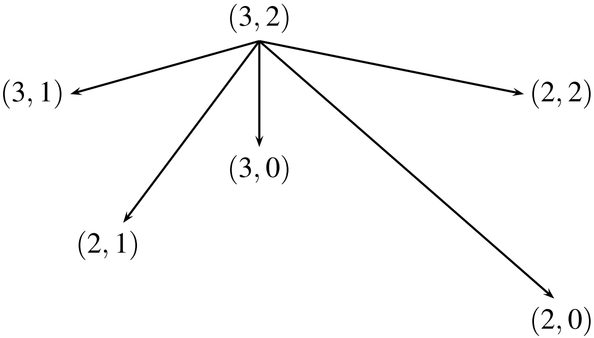

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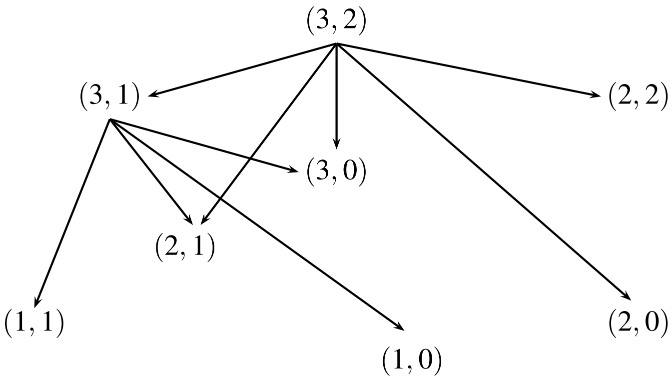

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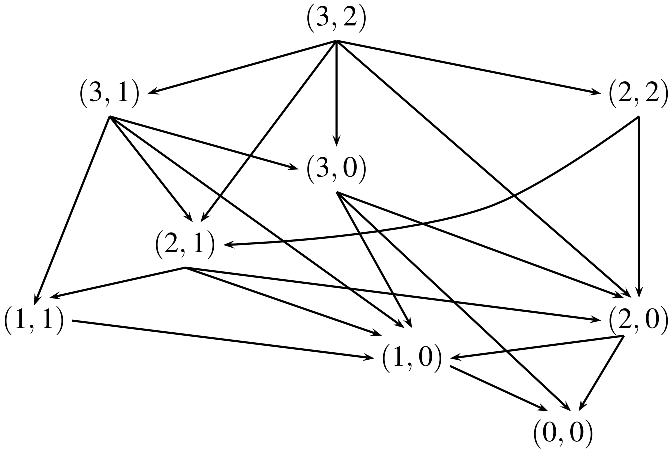

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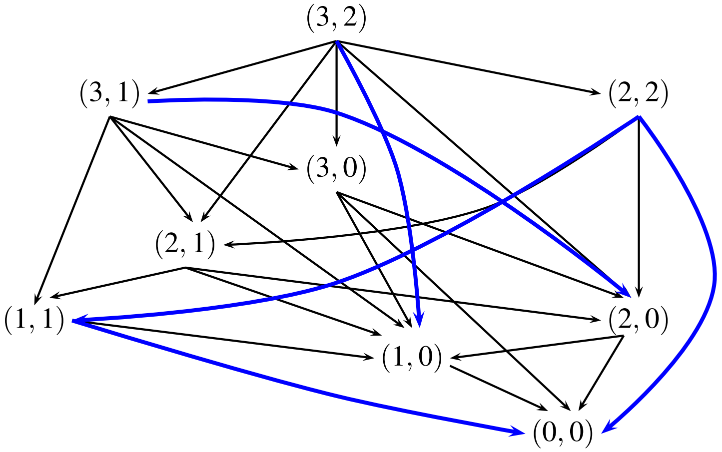

Impartial Games

Definition

A position is a $\mathcal{P}$ *position* for the player about to make a move if the $\mathcal{P}$revious player can force a win (that is, the player about to make a move is in a losing position). The position is a $\mathcal{N}$ *position* if the $\mathcal{N}$ext player (the player about to make a move) can force a win.

For impartial games, there are only two outcome classes for any position, namely **winning position** ($\mathcal{N}$ position) or **losing position** ($\mathcal{P}$ position). The set of losing positions is denoted by $\mathcal{L}$.

Recursive labeling

To find out whether Player I has a winning strategy, we label the nodes of the game graph **recursively** as follows:

- Sinks of the game graph are always losing ($\mathcal{P}$) positions.

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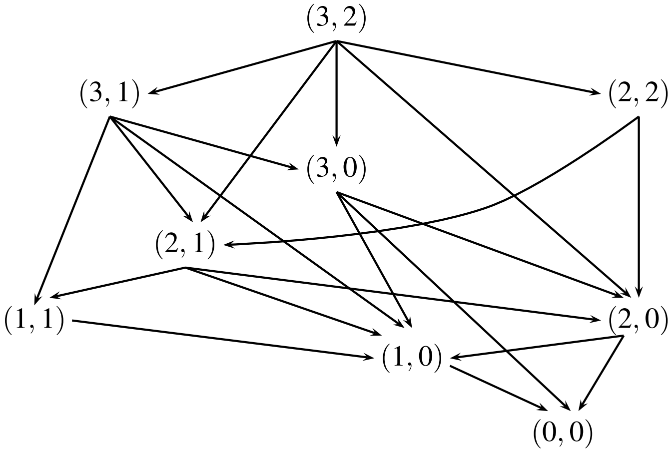
The label of the starting position of the game then tells whether Player I ($\mathcal{N}$) or Player II ($\mathcal{P}$) has a winning strategy.

Is $(3, 2)$ winning or losing for Nim ?



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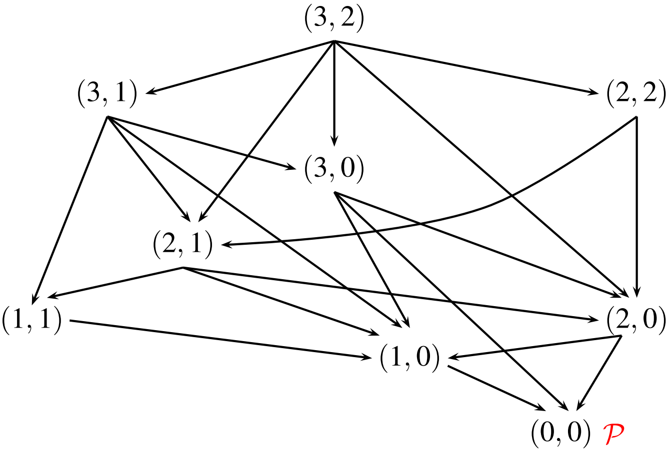

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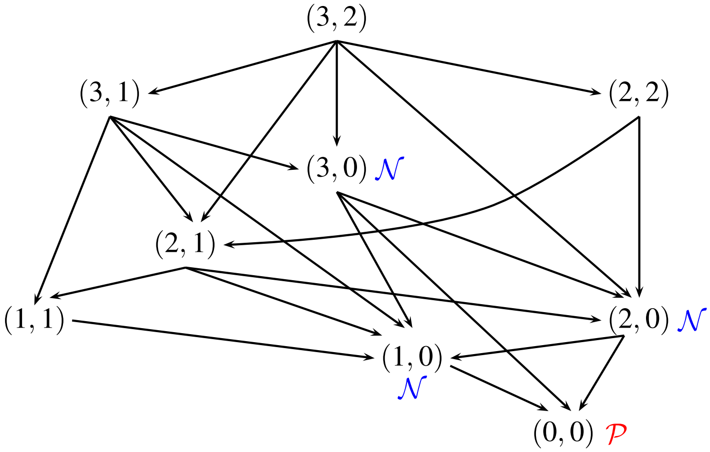

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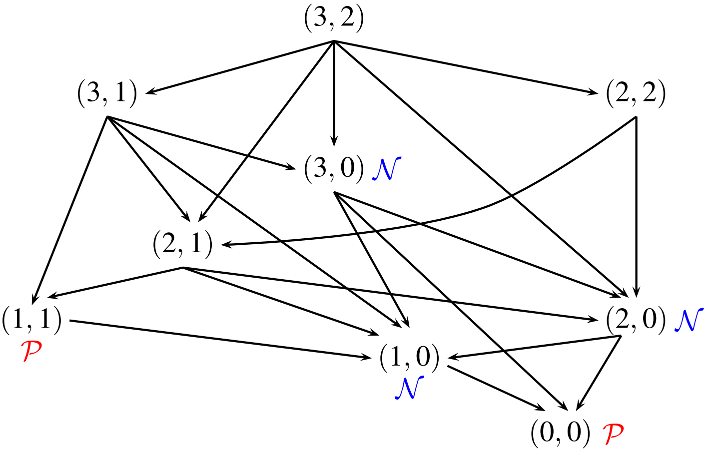

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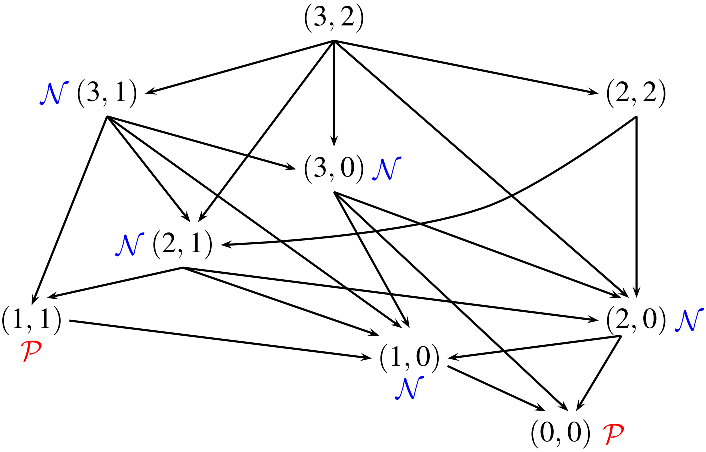

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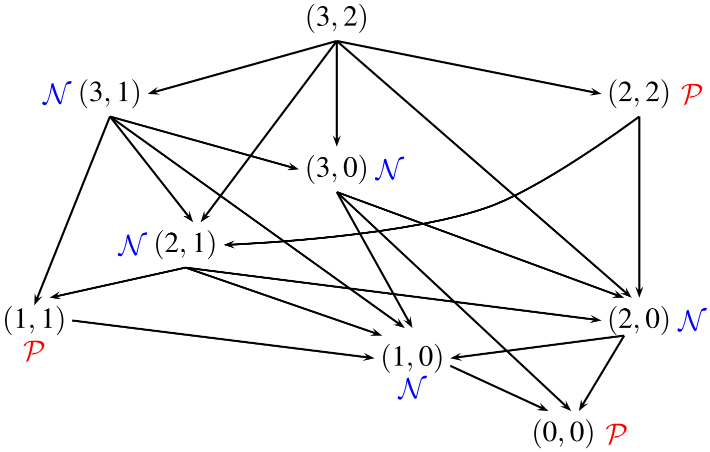

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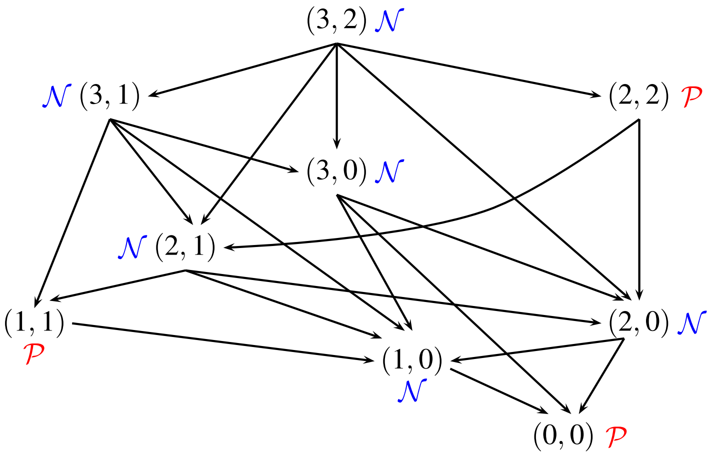

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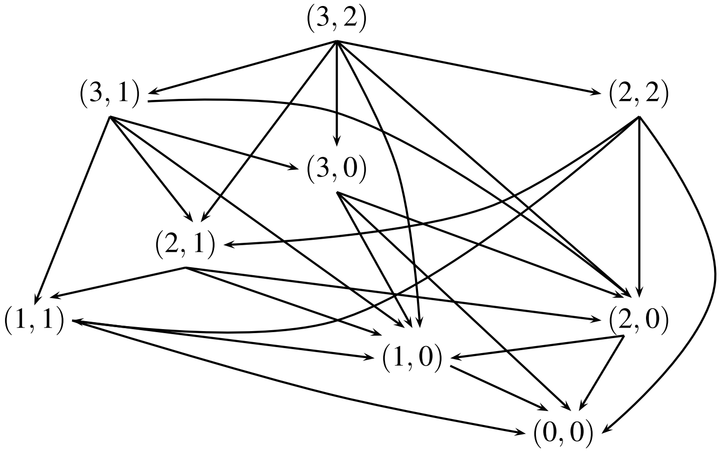

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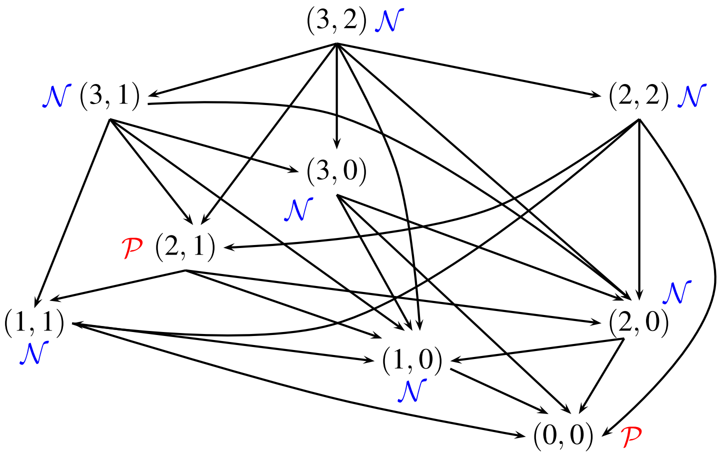

Take home lesson

- There is no legal move from a losing position to another losing position
- There is a recursive way to determine whether a position is losing or winning
- One can define a recursive function, the Grundy function, whose value is zero for a losing position, and positive for a winning position.
- Using a computer program, one can then obtain losing positions and guess a pattern for the losing positions.

An important tool

Theorem

Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets A and B with the properties:

I. *every option of a position in A is in B;*
II. *every position in B has at least one option in A; and*
III. *the final positions are in A.*

Then $A = \mathcal{L}$ *and* $B = \mathcal{W}$.

Proof strategy

- Obtain a candidate set S for the set of losing positions $\mathcal{L}$
- Show that any move from a position $\mathbf{p} \in S$ leads to a position $\mathbf{p}' \notin S$ (I)
- Show that for every position $\mathbf{p} \notin S$, there is a move that leads to a position $\mathbf{p}' \in S$ (II)

Often (as is the case for Nim and Wythoff), $(0, 0, \ldots, 0)$ is the only final position and it is easy to see that (III) is satisfied.

How to win in Nim

Definition

The *digital sum* $a \oplus b \oplus \cdots \oplus k$ of of integers $a, b, \ldots, k$ is obtained by translating the values into their binary representation and then adding them without carry-over.

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Example

The digital sum $12 \oplus 13 \oplus 7$ equals 6:

12	1	1	0	0
13	1	1	0	1
7		1	1	1
	0	1	1	0

How to win in Nim

Theorem

For the game of Nim, the set of losing positions is given by $\mathcal{L} = \{(p_1, p_2, \ldots, p_n) \mid p_1 \oplus p_2 \oplus \cdots \oplus p_n = 0\}$.

How to win in Wythoff

Let $\varphi = \frac{1+\sqrt{5}}{2}$. Then the set of losing positions is given by

$$\mathcal{L} = \{(\lfloor n \cdot \varphi \rfloor, \lfloor n \cdot \varphi \rfloor + n) | n \geq 0\}$$

Elements $(a_n, b_n) \in \mathcal{L}$ can be created recursively as follows:

- For a_n, find he smallest positive integer not yet used for a_i and b_i, $i < n$.
- $b_n = a_n + n$.

n	0	1	2	3	4	5
a_n						
b_n						

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n	0	1	2	3	4	5
a_n	0	1	3	4	6	8
b_n	0	2	5	7	10	13

Theorem

For the game of Wythoff, for any given position (a, b) there is exactly one losing position of each of the forms (a, y), (x, b), $(z, z + (b - a))$ for some $x \geq 0$, $y \geq 0$, and $z \geq 0$.

This structural result can be visualized as follows:

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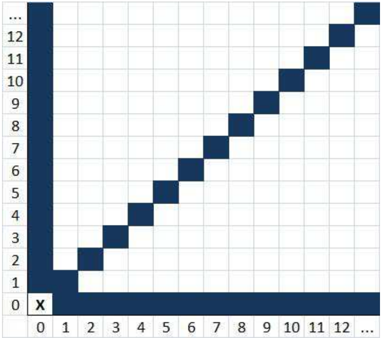

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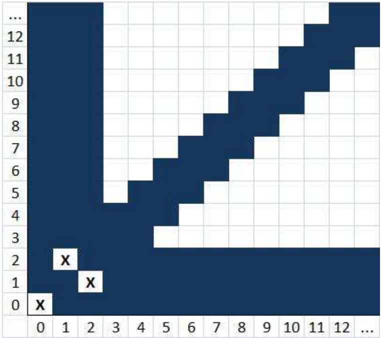

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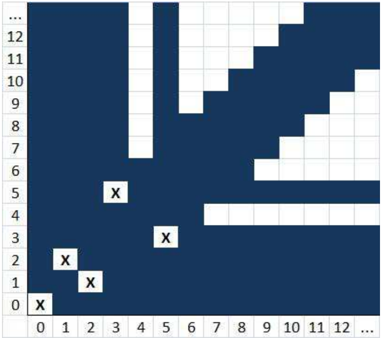

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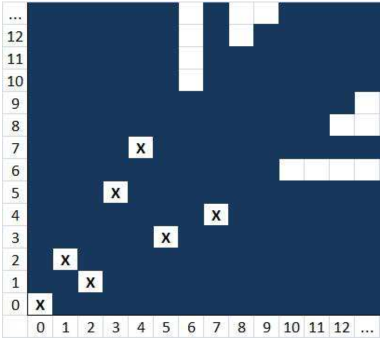

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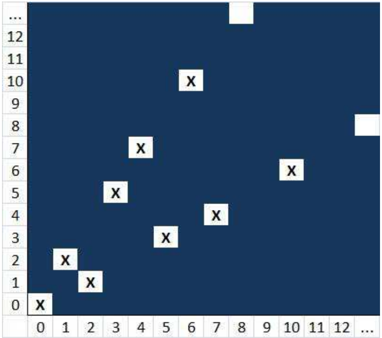

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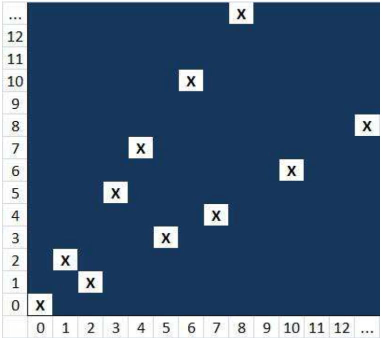

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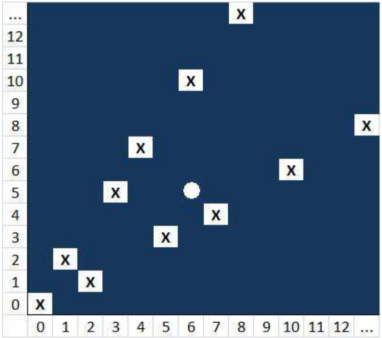
This structural result can be visualized as follows: $(a, b) = (6, 5)$



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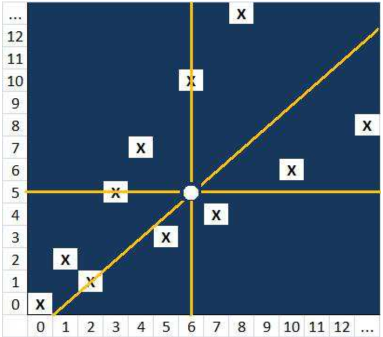

Losing positions: $(6, 10)$, $(3, 5)$, and $(2, 1)$.

Generalization of Wythoff to *n* stacks

Wythoff: Take any number of tokens from **one** stack OR select the **same** number of tokens from **both** stacks

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- take the **same** number of tokens from any **TWO** stacks
- take the **same** number of tokens from any non-empty **SUBSET** of stacks

Generalized Wythoff on n stacks

Let $B \subseteq \mathcal{P}(\{1, 2, 3, \ldots, n\})$ with the following conditions:

1. $\varnothing \notin B$
2. $\{i\} \in B$ for $i = 1, \ldots, n$.

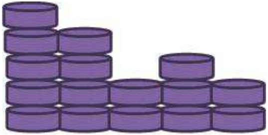
A legal move in generalized Wythoff $\mathcal{GW}_n(B)$ on n stacks induced by B consists of:

- Choose a set $A \in B$
- Remove the same number of tokens from each stack whose index is in A

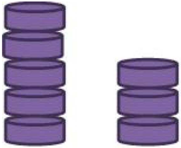
The first player who cannot move loses.

Examples

- **Nim**: Select one of the n stacks, take at least one token



- **Wythoff**: Either take any number of tokens from **one** stack OR select the **same** number of tokens from both stacks



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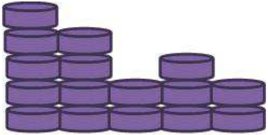
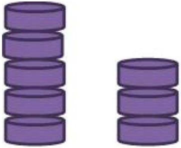
  $B = \{\{1\}, \{2\}, \ldots, \{n\}\}$

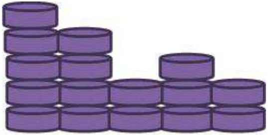


- **Wythoff**: Either take any number of tokens from **one** stack OR select the **same** number of tokens from both stacks

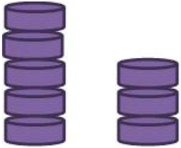


Examples

- **Nim**: Select one of the n stacks, take at least one token
  $B = \{\{1\}, \{2\}, \dots, \{n\}\}$



- **Wythoff**: Either take any number of tokens from **one** stack OR select the **same** number of tokens from both stacks



$B = \{\{1\}, \{2\}, \{1, 2\}\}$

$\overrightarrow{e}_i = i^{\text{th}}$ unit vector; $\overrightarrow{e}_A = \sum_{i \in A} \overrightarrow{e}_i$

Conjecture

In the game of generalized Wythoff $\mathcal{GW}_n(B)$, for any position $\overrightarrow{p} = (p_1, p_2, \ldots, p_n)$ and any $A = \{i_1, i_2, \ldots, i_k\} \in B$, there is a unique losing position of the form $\overrightarrow{p} + m \cdot \overrightarrow{e}_A$, where $m \geq -\min_{i \in A}\{p_i\}$.

Theorem

The conjecture is true for $|A| \leq 2$, that is, for any given position we can find a losing position for which only one or two of the stack heights are changed.

Example

$\mathcal{GW}_3(\{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\})$ - three stacks, with play on either a single or a pair of stacks. $\overrightarrow{p} = (11,17,20)$

A	$\tilde{p} \in \mathcal{L}$	$=$	$\overrightarrow{p}$	$+$	$m \cdot \overrightarrow{e}_A$
$\{1\}$	$(26,17,20)$	$=$	$(11,17,20)$	$+$	$15 \cdot (1,0,0)$
$\{2\}$	$(11,31,20)$	$=$	$(11,17,20)$	$+$	$14 \cdot (0,1,0)$
$\{3\}$	$(11,17,36)$	$=$	$(11,17,20)$	$+$	$16 \cdot (0,0,1)$
$\{1,2\}$	$(19,25,20)$	$=$	$(11,17,20)$	$+$	$8 \cdot (1,1,0)$
$\{1,3\}$	$(1,17,10)$	$=$	$(11,17,20)$	$-$	$10 \cdot (1,0,1)$
$\{2,3\}$	$(11,35,38)$	$=$	$(11,17,20)$	$+$	$18 \cdot (0,1,1)$

Example

$B_1 = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}); B_2 = B_1 \cup \{1,2,3\}$

$$\overrightarrow{p} = (11, 17, 20)$$

A	$\tilde{p}_1$	$\tilde{p}_2$
$\{1\}$	$(26, 17, 20)$	$(40, 17, 20)$
$\{2\}$	$(11, 31, 20)$	$(11, 1, 20)$
$\{3\}$	$(11, 17, 36)$	$(11, 17, 27)$
$\{1,2\}$	$(19, 25, 20)$	$(7, 13, 20)$
$\{1,3\}$	$(1, 17, 10)$	$(8, 17, 17)$
$\{2,3\}$	$(11, 35, 38)$	$(11, 12, 15)$
$\{1,2,3\}$	—	$(15, 21, 24)$

Proof for $|A| = 1$.

To show: For any position $(p_1, p_2, \ldots, p_n)$ there exists a unique position $(x, p_2, \ldots, p_n) \in \mathcal{L}$.

Uniqueness: Assume there are at least two positions of this form, $\tilde{p}_1 = (x, p_2, \ldots, p_n)$ and $\tilde{p}_2 = (y, p_2, \ldots, p_n)$, both in $\mathcal{L}$, with $x > y$. Then there exists a legal move from a losing position to a losing position (which is not possible) by taking $x - y$ tokens from the first stack of $\tilde{p}_1 = (x, p_2, \ldots, p_n)$. This is an allowed move as B always contains the singletons.

□

Proof for $|A| = 1$ continued.

Existence: Assume all positions of the form $\mathbf{p} = (x, p_2, \ldots, p_n)$ are winning positions. Upper bound on the number of moves from $\mathbf{p}$:

- $2^n - 1$ ways to choose the stacks to play on
- $\max_{i=2\ldots n} p_i$ different choices for number of tokens
- Let $M = (2^n - 1)(\max_{i=2\ldots n} p_i)$.

Now consider the $M + 1$ positions

$$\begin{array}{c}(0, p_2, \ldots, p_n)\\(1, p_2, \ldots, p_n)\\\vdots\\(M, p_2, \ldots, p_n)\end{array}$$

□

Proof for $|A| = 1$ continued.

$(i, p_2, \ldots, p_n) \in \mathcal{W}$ implies that there is at least one move $\mathbf{t_i}$ from $(i, p_2, \ldots, p_n)$ to a losing position $\mathbf{q_i}$.

$$\begin{array}{cccccc} (0, p_2, \ldots, p_n) & + & \mathbf{t_0} & = & \mathbf{q_0} \in \mathcal{L} \\ (1, p_2, \ldots, p_n) & + & \mathbf{t_1} & = & \mathbf{q_1} \in \mathcal{L} \\ \vdots & & & & \\ (M, p_2, \ldots, p_n) & + & \mathbf{t_M} & = & \mathbf{q_M} \in \mathcal{L} \end{array}$$

□

Proof for $|A| = 1$ continued.

By the pigeon hole principle, there must be a repeated move, say $\mathbf{t}$, yielding

$$\mathbf{q}_i = (i, p_2, \ldots, p_n) - \mathbf{t} = (i - t_1, p_2 - t_2, \ldots, p_n - t_n) \in \mathcal{L}$$

$$\mathbf{q}_j = (j, p_2, \ldots, p_n) - \mathbf{t} = (j - t_1, p_2 - t_2, \ldots, p_n - t_n) \in \mathcal{L}$$

But we already saw that this is not possible, and so there must be a losing position of the form $(x, p_2, \ldots, p_n)$. The proof easily applies to any set $A = \{i\}$. □

Note: What we have proved is that from any position we can "see" a losing position in any direction parallel to one of the axes of $\mathbb{R}^n$.

Proof for $|A| = 2$

- Proof is much more complicated
- We define the notion of a Wythoff set (a set that generalizes the properties of the set of losing positions constructed recursively for Wythoff)
- Uses a theorem about the interplay between the cardinalities of a sequence of two increasing sets and their accumulated sizes (= sums of their respective elements)
- Does not yet seem to generalize to $|A| > 2$.

Thank You!

Slides available from

`http://www.calstatela.edu/faculty/sheubac`

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Mex

Definition

The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set. It is denoted by $\text{mex}\{a, b, c, \ldots, k\}$.

Example

$$\text{mex}\{1, 4, 5, 7\} =$$
$$\text{mex}\{0, 1, 2, 6\} =$$

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$$\text{mex}\{1,4,5,7\} = 0$$
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The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set. It is denoted by $\text{mex}\{a, b, c, \ldots, k\}$.

Example

$$\text{mex}\{1,4,5,7\} = 0$$
$$\text{mex}\{0,1,2,6\} = 3$$

The Grundy Function

Definition

The Grundy function $\mathcal{G}(\mathbf{p})$ of a position $\mathbf{p}$ is defined recursively as follows:

- $\mathcal{G}(\mathbf{p}) = 0$ for any final position $\mathbf{p}$.
- $\mathcal{G}(\mathbf{p}) = \text{mex}\{\mathcal{G}(\mathbf{q}) | \mathbf{q} \text{ is an option of } \mathbf{p}\}$.

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Theorem

For a finite impartial game, $\mathbf{p}$ *belongs to class* $\mathcal{P}$ *if and only if* $\mathcal{G}(\mathbf{p}) = 0$.