Sizes of Graphs with Fixed Orders and Spans for Circular-Distance-Two Labelings *

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Abstract

A k-circular-distance-two labeling (or k-c-labeling) of a simple graph G is a vertex-labeling, using the labels $0, 1, 2, \cdots, k-1$, such that the "circular difference" (mod k) of the labels for adjacent vertices is at least two, and for vertices of distance-two apart is at least one. The σ-number, $\sigma(G)$, of a graph G is the minimum k of a k-c-labeling of G. For any given positive integers n and k, let $\mathcal{G}^\sigma(n,k)$ denote the set of graphs G on n vertices and $\sigma(G) = k$. We determine the maximum size (number of edges) and the minimum size of a graph $G \in \mathcal{G}^\sigma(n,k)$. Furthermore, we prove that for any value p between the maximum and the minimum size, there exists a graph $G \in \mathcal{G}^\sigma(n,k)$ of size p. These results are analogues of the ones by Georges and Mauro [4] on distance-two labelings.

Keywords. Vertex-labeling, circular difference, circular-distance-two labeling, distance-two labeling.

1 Introduction

Originated from the channel assignment problem introduced by Hale [7], distance-two labeling was first introduced and studied by Griggs and Yeh [6]. Given a graph

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G, the *distance* between two vertices u and v, denoted as $\mathrm{d}_G(u,v)$, is the length (number of edges) of a shortest path between u and v. An $L(2,1)$-labeling of G is a function, $f : V(G) \to \{0, 1, 2, \cdots\}$, such that $|f(u) - f(v)| \geq 2$ if $\mathrm{d}_G(u,v) = 1$, and $|f(u) - f(v)| \geq 1$ if $\mathrm{d}_G(u,v) = 2$. The span of an $L(2,1)$-labeling f is the difference of the maximum and minimum labels used by f. The λ-*number* of G, $\lambda(G)$, is defined as the minimum span among all $L(2,1)$-labelings of G.

We consider a variation of the $L(2,1)$-labeling by using a different measurement. For a positive integer k, a k-*circular-labeling* (or k-c-labeling for short) of a graph G is a function, $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$, such that:

$$|f(u) - f(v)|_k \geq \begin{cases} 2, & \text{if } \mathrm{d}_G(u,v) = 1; \\ 1, & \text{if } \mathrm{d}_G(u,v) = 2. \end{cases} \tag{$*$}$$

where $|x|_k := \min\{|x|, k - |x|\}$ is the *circular difference* modulo k. The σ-*number* of G, $\sigma(G)$, is the minimum k of a k-c-labeling of G.

The notion of using circular difference in the channel assignment problem was first introduced by ven den Heuvel, Leese and Shepherd [8]. For given positive integers d and $b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_d$, a k-*circular-distance-*d *labeling* is a function f which assigns to each vertex in G a label from the set $\{0, 1, 2, \cdots, k-1\}$ such that $|f(u) - f(v)|_k \geq b_i$ if $\mathrm{d}_G(u,v) = i$. In [8], the authors determined the minimum value k of a k-circular-distance-3 labeling for infinite triangular lattices and for infinite square lattices.

We denote the *size* (number of edges) and *order* (number of vertices) of a graph G by $\varepsilon(G)$ and $\nu(G)$, respectively. Fix positive integers n and k, let $\mathcal{G}^\sigma(n,k)$ denote the set of graphs G on n vertices and $\sigma(G) = k$. Let $\mathrm{m}^\sigma(n,k)$ and $\mathrm{M}^\sigma(n,k)$ denote, respectively, the minimum size and the maximum size of a graph $G \in \mathcal{G}^\sigma(n,k)$. Note that the σ-number of a graph on n vertices is at most $2n$ and can never be 2 or 3. Thus throughout the article, we assume $k = 1$ or $4 \leq k \leq 2n$. The main results of this article are:

Theorem 1.1 *Let n and k be positive integers, $k \le 2n$. Then*

$$\mathrm{m}^{\sigma}(n,k) = \begin{cases} 0, & \text{if } k = 1; \\ k-3, & \text{if } 4 \le k \le n+2; \\ \frac{(k-3n)(n-k+1)}{2}, & \text{if } k \text{ is even and } n+2 < k \le \frac{6n+6}{5}; \\ \frac{k(k-2)}{8}, & \text{if } k \text{ is even, } k > n+2 \text{ and } k > \frac{6n+6}{5}; \\ \frac{(k-3n)(n-k+1)}{2}, & \text{if } k \text{ is odd and } n+2 < k \le \frac{6n+9}{5}; \\ \frac{(k+3)(k-3)}{8}, & \text{if } k \text{ is odd, } k > n+2 \text{ and } k > \frac{6n+9}{5}. \end{cases}$$

Theorem 1.2 *Let n and k be positive integers with $n = qk + r$ for some integers q and $0 \le r < k$. Then*

$$\mathrm{M}^{\sigma}(n,k) = \begin{cases} 0, & \text{if } k = 1; \\ \binom{n}{2} - 2n + k, & \text{if } n+1 \le k \le 2n; \\ q\left(\binom{k}{2} - k\right) + \binom{r}{2}, & \text{if } 2r \le k \text{ and } 4 \le k \le n; \\ q\left(\binom{k}{2} - k\right) + \binom{r}{2} - 2r + k, & \text{if } 2r > k \text{ and } 4 \le k \le n. \end{cases}$$

Denote the minimum size and the maximum size of a graph with order n and λ-number k by $\mathrm{m}(n,k)$ and $\mathrm{M}(n,k)$, respectively. Georges and Mauro [4] obtained the formulas of $\mathrm{m}(n,k)$ and $\mathrm{M}(n,k)$ in terms of n and k. To prove Theorems 1.1 and 1.2, we make use of these Georges-Mauro formulas and results in [9] on connections between the σ-number and the λ-number.

In addition, for any given n and k, we characterize the graphs of minimum sizes in $\mathcal{G}^{\sigma}(n,k)$, and prove that for any integer p where $\mathrm{m}^{\sigma}(n,k) \le p \le \mathrm{M}^{\sigma}(n,k)$, there exists a graph $G \in \mathcal{G}^{\sigma}(n,k)$ of size p.

2 Preliminaries and notation

The *path covering number* of a graph G, $\mathrm{p_v}(G)$, is the minimum number of vertex disjoint paths covering $V(G)$. The *complement* of a graph G is denoted by G^c. Georges, Mauro and Whittlesey [5] proved an interesting result relating the path covering number of G^c and the λ-number for any graph G.

Theorem 2.1 *[5] Let G be a graph on n vertices. Then*

$$\lambda(G) \begin{cases} \le n-1, & \text{if } G^c \text{ has a Hamilton path;} \\ = n + \mathrm{p_v}(G^c) - 2, & \text{otherwise.} \end{cases}$$

The path covering number of G^c and the σ-number of G are also closely related. The following result was given in [9].

Theorem 2.2 *[9] Given a graph G on n vertices, then*

$$\sigma(G)\begin{cases} \leq n, & \text{if } G^c \text{ is Hamiltonian;} \\ = n + \mathrm{p_v}(G^c), & \text{if } G^c \text{ is not Hamiltonian.} \end{cases}$$

Combining Theorems 2.1 and 2.2, we have the following corollaries.

Corollary 2.3 *If G is a graph on n vertices and $\sigma(G) \geq n+2$, then $\sigma(G) = \lambda(G)+2$.*

Corollary 2.4 *If n and k are positive integers with $n+2 \leq k \leq 2n$, then $\mathrm{m}^\sigma(n,k) = \mathrm{m}(n,k-2)$ and $\mathrm{M}^\sigma(n,k) = \mathrm{M}(n,k-2)$.*

Note that in general, by definition, for any graph G, $\sigma(G)$ is either $\lambda(G)+1$ or $\lambda(G)+2$, and each case is realizable (cf. [9]). However, the problem of characterizing graphs into these two cases remains open.

Now we introduce several notation and definitions that will be used in later sections. For any positive integers n and k with $k \leq 2n$, let $\mathcal{G}^\sigma_{\mathrm{m}}(n,k)$ and $\mathcal{G}^\sigma_{\mathrm{M}}(n,k)$ denote, respectively, the set of graphs with the smallest and the largest sizes in $\mathcal{G}^\sigma(n,k)$. That is, for any $G \in \mathcal{G}^\sigma(n,k)$ we have $\mathrm{m}^\sigma(n,k) = \varepsilon(G)$ iff $G \in \mathcal{G}^\sigma_{\mathrm{m}}(n,k)$, and $\mathrm{M}^\sigma(n,k) = \varepsilon(G)$ iff $G \in \mathcal{G}^\sigma_{\mathrm{M}}(n,k)$.

Let L be a k-c-labeling of a graph G. Define:

$$\begin{cases} L_i := \{v \mid L(v) = i\} \text{ and } l_i := |L_i|, \ 0 \leq i \leq k-1; \\ H(L) := \{i \mid L_i = \emptyset\}; \\ G(L) := \{i \mid L_i = \emptyset \text{ and } l_{i-1} = l_{i+1} = 1\}. \end{cases}$$

All the indices above are taken under modulo k. If $i \in H(L)$ or $G(L)$, then i is called a *hole* or a *gap* of L, respectively.

A k-c-labeling L of a graph G is optimal if $k = \sigma(G)$. In this case, L is called a *σ-labeling*. An optimal $L(2,1)$-labeling is called a *λ-labeling*. A σ-labeling of G is *min-hole* if it has the least holes among all σ-labelings of G.

3 Proof of Theorem 1.1

We prove Theorem 1.1 by using Corollary 2.4 and the following result of Georges and Mauro [4]:

Theorem 3.1 *[4]*

$$\mathrm{m}(n,k)=\begin{cases} 0, & \text{if } k=0;\\ k-1, & \text{if } 2\le k\le n;\\ \frac{(k-3n+2)(n-k-1)}{2}, & \text{if } k \text{ is even and } n<k\le\frac{6n-4}{5};\\ \frac{k(k+2)}{8}, & \text{if } k \text{ is even, } n<k \text{ and } k>\frac{6n-4}{5};\\ \frac{(k-3n+2)(n-k-1)}{2}, & \text{if } k \text{ is odd and } n<k\le\frac{6n-1}{5};\\ \frac{(k-1)(k+5)}{8}, & \text{if } k \text{ is odd, } n<k \text{ and } k>\frac{6n-1}{5}. \end{cases}$$

(Proof of Theorem 1.1) The case as $k=1$ is trivial.

By Corollary 2.4, it remains to show the case that $4\le k\le n+1$. Hence, it suffices to prove $\mathrm{m}^\sigma(n,k)=k-3$ for all $4\le k\le n+1$.

Let $G=K_{1,k-3}\cup(n-k+2)K_1$, then $\sigma(G)=k$, so $\mathrm{m}^\sigma(n,k)\le k-3$. Suppose $\mathrm{m}^\sigma(n,k)\le k-4$. Let n_0 be the smallest n such that $n\ge k-1$ and $\mathrm{m}^\sigma(n,k)\le k-4$.

Let G be a graph in $\mathcal{G}^\sigma_{\mathrm{m}}(n_0,k)$. Then $\varepsilon(G)\le k-4$ and $\nu(G)=n_0\ge k-1\ge 3$, so G is not connected. Because $G\in\mathcal{G}^\sigma_{\mathrm{m}}(n_0,k)$, we have $G=G_1\cup xK_1$, for some $x\ge 2$ and G_1 a connected graph with $\sigma(G_1)=k$. Let $G'=G_1\cup(x-1)K_1$, then $\nu(G')=n_0-1$, $\varepsilon(G')=\varepsilon(G_1)=\varepsilon(G)\le k-4$ and $\sigma(G')=\sigma(G_1)=k$. By the minimality of n_0, one has $n_0=k-1$, which is impossible. For if $n_0=k-1$, then $\mathrm{m}^\sigma(k-2,k)\le\varepsilon(G')\le k-4$, contradicting the fact, by Corollary 2.4 and Theorem 3.1, that $\mathrm{m}^\sigma(n,n+2)=\mathrm{m}(n,n)=n-1$. Q.E.D.

Corollary 3.2 *If $4\le k\le 2n$, then $\mathrm{m}^\sigma(n,k)=\mathrm{m}(n,k-2)$ for all n.*

4 Proof of Theorem 1.2

The following result was proved by Georges and Mauro [4].

Theorem 4.1 *[4] Let $n = q(k+1) + r$, $q \in Z^+ \cup \{0\}$, $0 \le r \le k$. Then*

$$\mathrm{M}(n,k) = \begin{cases} 0, & \text{if } k = 0; \\ \lfloor \frac{n}{2} \rfloor, & \text{if } k = 2 \text{ and } n \ge 2; \\ \binom{n}{2} - 2n + k + 2, & \text{if } n-1 \le k \le 2n-2,\ k \ge 3; \\ q\left(\binom{k+1}{2} - k\right) + \binom{r}{2}, & \text{if } 2r - 2 \le k \text{ and } 3 \le k \le n-1; \\ q\left(\binom{k+1}{2} - k\right) + \binom{r}{2} - 2r + k + 2, & \text{if } k < 2r - 2 \text{ and } 3 \le k \le n-1. \end{cases}$$

Lemma 4.2 *If $n + 1 \le k \le 2n$, then $\mathrm{M}^\sigma(n,k) = \mathrm{M}(n, k-2)$.*

Proof. By Corollary 2.4, we only have to show the case for $k = n+1$. That is, to prove $\mathrm{M}^\sigma(n, n+1) = \binom{n}{2} - n + 1$.

Let $G \in \mathcal{G}^\sigma_{\mathrm{M}}(n, n+1)$. Then Theorem 2.2 implies that $\mathrm{p_v}(G^c) = 1$. Thus, $\varepsilon(G^c) \ge n-1$, so $\mathrm{M}^\sigma(n,k) = \varepsilon(G) \le \binom{n}{2} - n + 1$. To see $\mathrm{M}^\sigma(n,k) \ge \binom{n}{2} - n + 1$, let G be the complement of P_n (path on n vertices) and appeal to Theorem 2.2. Q.E.D.

By Theorem 4.1 and Lemma 4.2, the second case in Theorem 1.2 can be obtained directly. The remaining proof of Theorem 1.2 will rely on the following lemmas.

Lemma 4.3 *Let n and k be integers such that $n = qk + r$ for some $q \in Z^+$ and $0 \le r < k$. Then*

$$\mathrm{M}^\sigma(n,k) \ge \begin{cases} q\left(\binom{k}{2} - k\right) + \binom{r}{2}, & \text{if } 2r \le k \text{ and } 4 \le k \le n; \\ q\left(\binom{k}{2} - k\right) + \binom{r}{2} - 2r + k, & \text{if } 2r > k \text{ and } 4 \le k \le n. \end{cases}$$

Proof. It is enough to find graphs $G \in \mathcal{G}^\sigma(n,k)$ such that $\varepsilon(G)$ are as desired. Fix n and k where $n = qk + r$ for some $q \in Z^+$ and $0 \le r < k$. Define $G^* = G^*(n,k)$ by:

$$V(G^*) = V_0 \cup V_1 \cdots \cup V_{k-1} \cup U_0 \cup U_1 \cdots \cup U_{k-1},$$

where $V_i = \{v_{i,1}, v_{i,2}, \cdots, v_{i,q}\}$ for $0 \le i \le k-1$, and each U_i is defined by the following: (All indices are taken under modular k.)

(1) If $r = 0$, then $U_i = \emptyset$ for all i.

(2) If $r \ge 1$ and $2r \le k$, then $U_{2i} = \{u_{2i}\}$ for $i = 0, 1, 2, \cdots, r-1$, and $U_i = \emptyset$ for others.

(3) If $2r > k$, then $U_{2i} = \{u_{2i}\}$ for $0 \le i \le \lfloor \frac{k}{2} \rfloor - 1$, $U_{2i-1} = \{u_{2i-1}\}$ for $0 \le i \le r - \lfloor \frac{k}{2} \rfloor - 1$, and $U_i = \emptyset$ for others.

And the edge set of G^* has:

$$E(G^*) = \{v_{j,x}v_{w,x} : |j-w|_k \ge 2, 1 \le x \le q\} \cup \{u_j u_w : |j-w|_k \ge 2\}.$$

Indeed, $G^* \cong qC_k^c \cup B$, where C_k^c is the complement of the k-cycle, and $V(B) = \cup_{0 \le i \le k-1} U_i$. Define a labeling L^* on G^* by $L^*(v_{i,x}) = L^*(u_i) = i$, for all $0 \le i \le k-1$ and $1 \le x \le q$. It is easy to check that L^* is a k-c-labeling for G^*. So $\sigma(G^*) \le k$. On the other hand, it can be verified that $\sigma(C_k^c) = k$. Therefore $\sigma(G^*) = k$, implying $\mathrm{M}^\sigma(n,k) \ge \varepsilon(G^*)$. The result so follows by simple calculations of $\varepsilon(G^*)$. Q.E.D.

Lemma 4.4 *Suppose $G \in \mathcal{G}_{\mathrm{M}}^\sigma(n,k)$, and L is a σ-labeling for G. Then for all $|i-j|_k \ge 2$, the subgraph induced by $L_i \cup L_j$ is a matching with* $\min\{l_i, l_j\}$ *edges.*

Proof. Let L be a σ-labeling for some $G \in \mathcal{G}_{\mathrm{M}}^\sigma(n,k)$, then each vertex in L_i is adjacent to at most one vertex in L_j, for any j, $|i-j|_k \ge 2$. Without loss of generality, suppose $1 \le l_i \le l_j$ for some i, j, $|i-j|_k \ge 2$. It suffices to show that any vertex in L_i is adjacent to at least one vertex in L_j. Suppose to the contrary, there exists $v \in L_i$ such that v is not adjacent to L_j. Then since $l_i \le l_j$, there exists $u \in L_j$ such that u is not adjacent to L_i. Let $G' = G \cup \{uv\}$. Then $\nu(G') = n$, $\varepsilon(G') = \varepsilon(G) + 1$, and L is a k-c-labeling for G'. Thus $\sigma(G') = k$, contradicting the assumption that $G \in \mathcal{G}_{\mathrm{M}}^\sigma(n,k)$. Q.E.D.

The next lemma is trivial.

Lemma 4.5 *If L is a σ-labeling for a graph G, then L has no consecutive holes.*

It was proved in [9] that if L is a min-hole σ-labeling for a graph G and $h \in H(L)$, then $l_{h-1} = l_{h+1} > 0$; moreover, $G(L) = \emptyset$ if and only if $\sigma(G) \le \nu(G)$. These lead to the next lemma:

Lemma 4.6 *Suppose $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$, $k \le n$, and L is a min-hole k-c-labeling for G. If $h \in H(L)$, then $l_{h-1} = l_{h+1} \ge 2$.*

Lemma 4.7 *If $n \ge k$, then there exists a graph $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$ such that $|H(L)| < k/2$ for some min-hole σ-labeling L of G.*

Proof. By Lemma 4.5, $|H(L)| \le k/2$. Hence, we only have to consider the case that k is even and $k \ge 4$. Suppose there exists $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$ such that $|H(L)| = k/2$, where L a min-hole σ-labeling of G. By Lemma 4.3, it suffices to show that $\varepsilon(G) \le \varepsilon(G^*)$. For then $G^* \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$, so the result follows by taking G^* and L^*, where L^* is the same as defined in Lemma 4.3 (note that $|H(L^*)| = 0$).

Following Lemma 4.3, set $n = qk + r$. By Lemma 4.6, without loss of generality, we may assume that $l_0 = l_2 = l_4 \cdots = l_{k-2} = b$, so $n = bk/2 = qk + r$. Hence $(b - 2q)k = 2r$. Because $r \le k - 1$, it follows that $b - 2q = 0$ or 1. If $b - 2q = 0$, then $r = 0$. Hence $\varepsilon(G) = qk(k-2)/4 \le qk(k-3)/2 = \varepsilon(G^*)$ (since $k \ge 4$).

If $b - 2q = 1$, then $k = 2r$. By Lemma 4.4, we get $\varepsilon(G) = \frac{qk(r-1)}{2} + \binom{r}{2} \le \frac{qk(k-3)}{2} + \binom{r}{2} = \varepsilon(G^*)$ (since $r \ge 2$). Q.E.D.

Lemma 4.8 *If $n \ge k$, then $\mathrm{M}^{\sigma}(n,k) < \mathrm{M}^{\sigma}(n,k+1)$.*

Proof. By Lemma 4.7, there exist a graph $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$ and a σ-labeling L of G such that $|H(L)| < k/2$. That is, there exist $u, v \in V(G)$, $uv \notin E(G)$, such that $L(u) = i$ and $L(v) = i + 1 \pmod{k}$ for some i. Let G' be the graph such that $V(G') = V(G)$ and $E(G') = E(G) \cup \{uv\}$. Because $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$, one has $\sigma(G') > k$. Define a labeling L' on G' by $L'(x) = L(x)$ if $L(x) \le i$; $L'(x) = L(x) + 1 \pmod{k+1}$ if $L(x) \ge i + 1$. Then L' is a $(k+1)$-c-labeling for G', so $\sigma(G') \le k + 1$. It then follows that $\sigma(G') = k + 1$, so $\mathrm{M}^{\sigma}(n,k+1) \ge \varepsilon(G') > \mathrm{M}^{\sigma}(n,k)$. Q.E.D.

Lemma 4.9 *If $n \ge k$, then there exists a graph $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$ such that G has an onto σ-labeling.*

Proof. Let L be a σ-labeling of a graph $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$ such that L has the least holes among all σ-labelings of graphs in $\mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$. By Lemma 4.7, $|H(L)| < k/2$. It suffices to show that L is onto. Suppose not, then there exists $h \in H(L)$ such that $l_{h+2} > 0$ or $l_{h-2} > 0$. Without loss of generality, suppose $l_{h+2} > 0$. Let $u \in L_{h+2}$.

By Lemma 4.6, $l_{h+1} = l_{h-1} \geq 2$. Let $x, x' \in L_{h-1}$ and $y, y' \in L_{h+1}$. By Lemma 4.4, without loss of generality, assume $xy, x'y' \in E(G)$ and $ux' \notin E(G)$ (since at most one of x and x' is adjacent to u). Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup \{x'u\} - \{x'y'\}$. Define a mapping L' on $V(G')$ by $L'(w) = L(w)$ if $w \neq x'$; $L'(x') = h$. Then L' is a k-c-labeling for G'. Since $\varepsilon(G') = \varepsilon(G)$, by Lemma 4.8, $\sigma(G') = k$. Hence $G' \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$. This contradicts the minimality of L, since L' has fewer holes than L. Therefore L is onto. Q.E.D.

Lemma 4.10 *If $n \geq k$, then there exists $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$, such that $G = A \cup B$, where $A \cong C^c_k$.*

Proof. By Lemma 4.9, there exist a graph $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$ and a k-c-labeling L for G such that $H(L) = \emptyset$. So by Lemma 4.4, we can find a k-cycle A in G,

$$A = \begin{cases} v_0, v_2, v_4, \cdots v_{k-2}, v_1, v_{k-1}, v_3, v_5, \cdots, v_{k-3}, v_0, & \text{if } k \text{ is even;} \\ v_0, v_2, v_4, \cdots v_{k-1}, v_1, v_3, \cdots, v_{k-2}, v_0, & \text{if } k \text{ is odd,} \end{cases}$$

where $v_i \in L_i$ for each i. Note that if $n = k$ then it is trivial that such a cycle exists. If $n > k$, then by Lemma 4.8, one can show (by replacing some edges if necessary) that there exists some graph $G \in \mathcal{G}^{\sigma}_{\mathrm{m}}(n,k)$ so that G contains such a cycle.

Now, it is enough to show that there exists $G' \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$ such that $G' = A \cup B$ (where A is the same as defined in the above) and $A \cong C^c_k$. For the latter part, it suffices to prove $v_i v_j \in E(G')$ for all $|i-j|_k \geq 2$.

Suppose $v_i v_j \notin E(G)$ for some $|i-j|_k \geq 2$. Then by Lemma 4.4, at least one of the following two occurs:

(a) $v_j x \in E(G)$ for some $x \in L_i$,

(b) $v_i y \in E(G)$ for some $y \in L_j$.

Let G' be the graph such that $V(G') = V(G)$. If (a) occurs only (the case for (b) occurs only is similar), let $E(G') = E(G) \cup \{v_i v_j\} - \{v_j x\}$; if both (a) and (b) occur, let $E(G') = E(G) \cup \{v_i v_j, xy\} - \{v_j x, v_i y\}$. In any of these two cases, applying the same labeling L on G' guarantees that $\sigma(G') \leq k$. Hence, by Lemma 4.8, $\sigma(G') = k$. This implies $G' \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n, k)$. Continuing this process, eventually, we obtain a graph $G' \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n, k)$ such that $G' = A \cup B$ and $A \cong C^c_k$. Q.E.D.

(Proof of Theorem 1.2) By Lemmas 4.2 and 4.3, it suffices to show that $\mathrm{M}^{\sigma}(n, k) \leq \varepsilon(G^*)$, if $n \geq k$. We prove this by induction on q.

Initial step: Let $q = 1$, then $n = k + r$, $0 \leq r < k$. By Lemma 4.10, there exists $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n, k)$ such that $G = A \cup B$, $A \cong C^c_k$ and $\nu(B) = r$. This implies that

$$\begin{aligned}
\mathrm{M}^{\sigma}(n,k) &= \varepsilon(A) + \varepsilon(B) \\
&= k(k-3)/2 + \varepsilon(B) \\
&\leq k(k-3)/2 + \mathrm{M}^{\sigma}(r,k) && \text{(by Lemma 4.8)} \\
&\leq \begin{cases} k(k-3)/2 + \binom{r}{2} - 2r + k, \text{ if } k \leq 2r & \text{(by Lemma 4.2, Theorem 4.1)} \\ k(k-3)/2 + \binom{r}{2}, \text{ if } k > 2r & \text{(by definition)} \end{cases} \\
&= \varepsilon(G^*) && \text{(by Lemma 4.3)}
\end{aligned}$$

Inductive step: Suppose $n = qk + r$, $q \geq 2$, $0 \leq r < k$. By Lemma 4.10, there exists $G \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n, k)$ such that $G = A \cup B$, $A \cong C^c_k$ and $\nu(B) = (q-1)k + r$. According to the inductive hypothesis, $\varepsilon(B) \leq \varepsilon(G^*(n-k, k))$. Hence $\mathrm{M}^{\sigma}(n, k) = \varepsilon(G) = k(k-3)/2 + \varepsilon(B) \leq \varepsilon(G^*(n, k))$. The proof is complete. Q.E.D.

5 All sizes are attainable

Fixing n and k, Georges and Mauro [4] characterized the graphs of minimum sizes on n vertices and $\lambda(G) = k$. Combining their results with Corollary 3.2, characterizations of graphs in $\mathcal{G}^{\sigma}_{\mathrm{m}}(n, k)$ can be obtained.

Let $G \in \mathcal{G}^{\sigma}(n, k)$, where $k \geq n + 3$. By Theorem 2.2, $\mathrm{p_v}(G^c) = k - n$. Let $c = k - n$. Define H_1 and H_2 by: $H_1 = ((c-1)K_1 \cup K_{n-c+1})^c$ and $H_2 = (jK_1 \cup K_{n+c-1-2j})^c \cup (j - c + 1)K_1$, where $j = \lfloor \frac{n+c-2}{2} \rfloor$.

Corollary 5.1 *Let H_1 and H_2 be defined as in the above. Then*

(1) for $4 \le k \le n+2$,

$$\mathcal{G}^{\sigma}_{\mathrm{m}}(n,k) = \begin{cases} \{K_{1,k-3} \cup (n-k+2)K_1\}, & \text{if } k \neq 6; \\ \\ \{K_{1,3} \cup (n-4)K_1, K^c_{1,3} \cup (n-4)K_1\}, & \text{if } k = 6. \end{cases}$$

(2) for $n+3 \le k \le 2n$,

$$\mathcal{G}^{\sigma}_{\mathrm{m}}(n,k) = \begin{cases} \{H_1\}, & \text{if } k \text{ is even and } k < \frac{6n+6}{5}, \text{ or } k \text{ is odd and } k < \frac{6n+9}{5}; \\ \\ \{H_1, H_2\}, & \text{if } k = \frac{6n+6}{5} \text{ or } k = \frac{6n+9}{5}; \\ \\ \{H_2\}, & \text{otherwise.} \end{cases}$$

Proof. The results for $n \le k-2$ follow directly from Theorem 3.6 in [4] and Corollary 3.2.

Suppose $n \ge k-1$. Let $G \in \mathcal{G}^{\sigma}_{\mathrm{m}}(n,k)$. By Theorem 1.1, $\varepsilon(G) = k-3 \le \nu(G)-2$. So G is not connected. Because $G \in \mathcal{G}^{\sigma}_{\mathrm{m}}(n,k)$, we have $G = G_1 \cup xK_1$ for some $x \ge 1$ and some connected graph G_1 such that $\varepsilon(G_1) = k-3$, $\sigma(G') = k$, and $G_1 \in \mathcal{G}^{\sigma}_{\mathrm{m}}(n-x,k)$. Hence $\varepsilon(G_1) \ge \nu(G_1)-1$, so $\nu(G_1) \le k-2$. Thus the results hold for G_1, and so hold for G. This completes the proof. Q.E.D.

Georges and Mauro [4] proved that all sizes between $\mathrm{m}(n,k)$ and $\mathrm{M}(n,k)$ are attained by some graphs with order n and λ-number k. Analogous results also hold for circular-distance-two labelings.

Theorem 5.2 *Given n and k, $4 \le k \le 2n$, then for any x where $\mathrm{m}^{\sigma}(n,k) \le x \le \mathrm{M}^{\sigma}(n,k)$, there exists a graph $G \in \mathcal{G}^{\sigma}(n,k)$ such that $\varepsilon(G) = x$.*

Proof. For any n and k, $4 \le k \le 2n$, it is enough to find graphs $G \in \mathcal{G}^{\sigma}_{\mathrm{m}}(n,k)$ and $G' \in \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$ such that G is a subgraph of G'. We consider two cases:

$\boxed{\text{Case 1}}$ $4 \le k \le n$: By the proof of Theorem 1.2, it suffices to find $G \in \mathcal{G}^{\sigma}_{\mathrm{m}}(n,k)$ such that $G \subseteq G^*$ (G^* is defined in Lemma 4.3). Let $n = qk+r$, $0 \le r < k$. Define a subgraph G_m of G^* by: $V(G_m) = V(G^*)$ and $E(G_m) = \{(v_{0,1}v_{i,1}) : 2 \leq i \leq k-2\}$. Then $G_m \cong K_{1,k-3} \cup (n-k+2)K_1$; by Corollary 5.1, $G_m \in \mathcal{G}^{\sigma}_{\mathrm{m}}(n,k)$.

$\boxed{\text{Case 2}}$ $k \geq n+1$: By Theorem 2.2, for any $G \in \mathcal{G}^{\sigma}_{\mathrm{m}}(n,k)$, $\mathrm{p_v}(G^c) = k-n \geq 1$. By Theorem 1.2, $\mathrm{M}^{\sigma}(n,k) = \binom{n}{2} - 2n + k$, so $\mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$ contains all the graphs G such that $\nu(G) = n$, and G is the complement of $(k-n)$ disjoint paths. It then follows that for any $G \in \mathcal{G}^{\sigma}_{\mathrm{m}}(n,k)$, $G \subseteq G'$ for some $G' \subseteq \mathcal{G}^{\sigma}_{\mathrm{M}}(n,k)$. Q.E.D.

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