Contemporary Mathematics

The Grothendieck Group and the Extensional Structure of Noetherian Module Categories

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Abstract. For a left Noetherian ring R, the Gothendieck group $G_0(R)$ is universal for maps which respect short exact sequences from the category of left Noetherian R-modules to Abelian groups. There is a less well known monoid $M(R\text{-}\mathbf{Noeth})$ which has the analogous universal property with respect to maps into commutative monoids. In this paper the relationship between these two universal objects is studied leading to a new and more detailed description of the former.

There is a natural decomposition $G_0(R) \cong \mathbb{Z}^n \times \widetilde{G}_0(R)$ where n is the number of minimal prime ideals of R and $\widetilde{G}_0(R)$ is a group which we show can be embedded in $M(R\text{-}\mathbf{Noeth})$, roughly speaking, as those elements which are comparable to the image of R in $M(R\text{-}\mathbf{Noeth})$. This leads to a description of $\widetilde{G}_0(R)$ in terms of generators $\langle A\rangle$ for modules A of reduced rank zero and certain relations of the form $\langle U/U'\rangle = 0$ where $U' \subset U$ are isomorphic uniform left ideals of minimal prime factor rings of R. In particular, for a domain of Krull dimension 1, the generators of $\widetilde{G}_0(R)$ correspond to simple modules, and the relations correspond to the composition series of the modules R/Rx when $x \in R$ is irreducible.

1. Introduction

Let R be a left Noetherian ring and R-**Noeth** the category of Noetherian left R-modules. One of the tools used to study R-**Noeth** is the Grothendieck group. This group, written $G_0(R)$, is, by definition, the Abelian group generated by the symbols $\langle A\rangle$ for all $A \in R\text{-}\mathbf{Noeth}$, subject to the relations $\langle B\rangle = \langle A\rangle + \langle C\rangle$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence in R-**Noeth**. The group $G_0(R)$ has the universal property that, if Λ is a map from R-**Noeth** into an Abelian group which respects short exact sequences (4.4), then Λ factors through $G_0(R)$.

In this paper we take the view that $G_0(R)$ should be constructed in two stages: Firstly, construct a universal monoid, denoted $M(R\text{-}\mathbf{Noeth})$, for maps which respect short exact sequences from R-**Noeth** into commutative monoids. Secondly, construct a universal group, $G(M(R\text{-}\mathbf{Noeth}))$, for monoid homomorphisms from $M(R\text{-}\mathbf{Noeth})$ into Abelian groups. From the universal properties we have immediately that $G_0(R) \cong G(M(R\text{-}\mathbf{Noeth}))$. We consider that $M(R\text{-}\mathbf{Noeth})$ encodes

1991 *Mathematics Subject Classification.* Primary 16D70, 16D90, 16P40; Secondary 20M14.

the "extensional structure" of the category R-**Noeth**, which explains the title of this paper.

By definition, $G_0(R)$ is a group, and so is cancellative, meaning that $a+c=b+c$ implies $a=b$ for all elements $a,b,c\in G_0(R)$. One reason for studying $M(R\text{-}\mathbf{Noeth})$ is that $M(R\text{-}\mathbf{Noeth})$ has cancellation properties which are not part of its definition, and which are overlooked by enforcing cancellation by going directly to $G_0(R)$. Specifically, it is proved in [**2**, 5.1] that $M(R\text{-}\mathbf{Noeth})$ is strongly separative, meaning that $a+a=b+a$ implies $a=b$ for all $a,b\in M(R\text{-}\mathbf{Noeth})$.

Let N be the prime radical of R, and Q the Goldie quotient ring of R/N. Then $G_0(R)\cong G_0(R/N)$ (5.2), and, because $Q_{R/N}$ is flat, there is a surjective group homomorphism from $G_0(R/N)$ to $G_0(Q)$. Thus we have a surjection $\tau\colon G_0(R)\to G_0(Q)$. The ring Q is semisimple so $G_0(Q)\cong\mathbb{Z}^n$ where n is the number of isomorphism classes of simple Q modules, or equivalently, n is the number of minimal prime ideals of R. Since $G_0(Q)$ is projective, τ splits and we get a decomposition $G_0(R)\cong\mathbb{Z}^n\times\widetilde{G}_0(R)$ where $\widetilde{G}_0(R)\cong\ker\tau$.

The decomposition $G_0(R)\cong\mathbb{Z}^n\times\widetilde{G}_0(R)$ is also reflected in the structure of $M(R\text{-}\mathbf{Noeth})$. Specifically, we show that $\widetilde{G}_0(R)$ is embedded in $M(R\text{-}\mathbf{Noeth})$ as the set of elements of $M(R\text{-}\mathbf{Noeth})$ which are "comparable" with r, the image of R in $M(R\text{-}\mathbf{Noeth})$: More precisely, the elements of $\widetilde{G}_0(R)$ are in bijection with the subset $\{\equiv r\}$ of $M(R\text{-}\mathbf{Noeth})$ defined by

$$\{\equiv r\}=\{a\in M(R\text{-}\mathbf{Noeth})\mid r\le a\le r\},$$

where $\le$ is the preorder on $M(R\text{-}\mathbf{Noeth})$ defined by ($a\le b\iff\exists c$ such that $a+c=b$). This embedding is not a monoid or group homomorphism, but if we define the operation $\Box_r$ on $\{\equiv r\}$ by $a\,\Box_r b=c$ if $a+b=r+c$, then $\widetilde{G}_0(R)$ is isomorphic to $(\{\equiv r\},\ \Box_r)$. Interestingly, $M(R\text{-}\mathbf{Noeth})$ contains not just $\widetilde{G}_0(R)$, but also $\widetilde{G}_0(S)$ for every factor ring S of R (6.7).

This description of $\widetilde{G}_0(R)$ has the consequence (6.6) that

$$\widetilde{G}_0(R)=\{\langle A\rangle\mid A\in R\text{-}\mathbf{Noeth}\text{ has reduced rank zero}\}.$$

This enables us to derive a set of generators and relations for $\widetilde{G}_0(R)$ (7.5). For example, in one simple case we get (7.8):

Theorem 1.1. *Let R be a left Noetherian domain with Krull dimension* 1*, and $\mathbb{S}$ a set of representatives of the isomorphism classes of simple left R-modules. Then $\widetilde{G}_0(R)$ is the Abelian group with one generator $\langle S\rangle$ for each $S\in\mathbb{S}$, and relations $\langle S_1\rangle+\langle S_2\rangle+\ldots+\langle S_k\rangle=0$ whenever $S_1,S_2,\ldots,S_k\in\mathbb{S}$ are isomorphic to the composition series factors of R/Rx with $x\in R$ irreducible.*

Since R is a prime ring, we have $G_0(R)\cong\mathbb{Z}\times\widetilde{G}_0(R)$ in this case.

2. Commutative Monoids

All monoids in this paper will be commutative, so we will write $+$ for the monoid operation and 0 for the identity element of all monoids. We refer the reader to [**5**] and [**4**] for the standard concepts of monoid theory.

We collect here some notation we will need:

Notation 2.1. Let M be a monoid and $a,b\in M$.

- $a\le b\iff\exists\, c\in M$ such that $a+c=b$

- $a \ll b \iff a + b \leq b$
- $a \propto b \iff \exists\, n \in \mathbb{N}$ such that $a \leq nb$
- $a \equiv b \iff a \leq b$ and $b \leq a$
- $\{\equiv a\} = \{c \in M \mid c \equiv a\}$

The relation $\leq$ is a preorder on M. For the monoid $(\mathbb{Z}^+, +)$, the set of nonnegative integers, this preorder coincides with the usual order. If M is a group, then we have $a \leq b$ for all $a, b \in M$. So for the monoid $(\mathbb{Z}, +)$, the preorder $\leq$ is not the same as the usual order on the integers.

The relation $\ll$ is transitive, $\propto$ is a preorder, and $\equiv$ is a congruence.

A monoid M is **cancellative** if for all $a, b, c \in M$, $a+c = b+c$ implies $a = b$. As mentioned in the introduction, the monoid $M(R$-**Noeth**$)$, though not, in general, cancellative, has a weak form of cancellation, called strong separativity: A monoid M is **strongly separative** [**1**] [**2**, 2.3] if for all $a, b \in M$, $2a = a + b$ implies $a = b$.

We will often use strong separativity in one of the following forms.

Lemma 2.2. *Let M be a strongly separative monoid, $n \in \mathbb{N}$ and $a, b, c \in M$.*

(1) $a + nc = b + nc$ and $c \propto a \implies a = b$
(2) $a + nc = b + nc \implies a + c = b + c$
(3) $a + nc \leq b + nc$ and $c \propto a \implies a \leq b$
(4) $a + nc \leq b + nc \implies a + c \leq b + c$

Proof. We prove first that $a + na = b + na$ implies $a = b$. The $n = 0$ case is trivial, so we will assume the claim is true for some $n \in \mathbb{Z}^+$ and consider the equation $a+(n+1)a = b+(n+1)a$. Adding na to both sides we get $2a+n(2a) = (a+b)+n(2a)$, so from the induction hypothesis, $2a = a + b$. Since M is strongly separative this implies $a = b$, completing the induction.

(1) Since $c \propto a$, there is some $m \in \mathbb{N}$ and $x \in M$ such that $c + x = ma$. Then $a + mna = a + nc + nx = b + nc + nx = b + mna$. By the above claim, this implies $a = b$.
(2) This follows from 1 since $c \propto a + c$.
(3) Since $a+nc \leq b+nc$, there is some $x \in M$ such that $(a+x)+nc = b+nc$. We also have $c \propto a + x$, so from 1 we get $a + x = b$, that is, $a \leq b$.
(4) This follows from 3 since $c \propto a + c$.

□

Given an arbitrary commutative monoid M, there is an associated Abelian group $G(M)$, called the **Grothendieck group** of M, [**9**, 1.1.3] and a monoid homomorphism $a \mapsto \langle\!\langle a \rangle\!\rangle$ from M to $G(M)$ with the following universal property: Given a monoid homomorphism $\lambda\colon M \to N$ where N is an Abelian group, there is a unique group homomorphism $\overline{\lambda}\colon G(M) \to N$ such that $\lambda(a) = \overline{\lambda}(\langle\!\langle a \rangle\!\rangle)$ for all $a \in M$.

Every element of $G(M)$ can be written in the form $\langle\!\langle a \rangle\!\rangle - \langle\!\langle b \rangle\!\rangle$ for some $a, b \in M$. For $a, b \in M$, we have $\langle\!\langle a \rangle\!\rangle = \langle\!\langle b \rangle\!\rangle$ if and only if there is some $c \in M$ such that $a + c = b + c$. Consequently, the monoid homomorphism $a \mapsto \langle\!\langle a \rangle\!\rangle$ is injective if and only if M is cancellative.

We write $G^+(M) = \{\langle\!\langle a \rangle\!\rangle \mid a \in M\}$ for the image of M in $G(M)$. $G^+(M)$ is a submonoid of $G(M)$. An easy calculation shows that $G(G^+(M)) \cong G(M)$, so that $G^+(M)$ determines $G(M)$.

The monoid $G^+(M)$ has its own universal property: Given a monoid homomorphism $\lambda\colon M \to N$ where N is cancellative, there is a unique monoid homomorphism $\overline{\lambda}\colon G^+(M) \to N$ such that $\lambda(a) = \overline{\lambda}(\langle\langle a\rangle\rangle)$ for all $a \in M$.

When the monoid M is strongly separative, the map from M to $G^+(M)$ may not be injective as it is if M were cancellative. Nonetheless this map will still be injective on elements of M which are "big" in the sense of the following definition: An element u of a monoid M is an **order unit** if $a \propto u$ for all $a \in M$.

Combining this definition, Lemma 2.2(1) and the fact that $\langle\langle a\rangle\rangle = \langle\langle b\rangle\rangle$ in $G(M)$ if and only if there is some $c \in M$ such that $a + c = b + c$, we get

LEMMA 2.3. *Let u be an order unit in a strongly separative monoid M and $a, b, c \in M$.*

(1) *a is an order unit $\iff$ $u \propto a$*
(2) *If a is an order unit and $a + c = b + c$, then $a = b$.*
(3) *If a is an order unit and $\langle\langle a\rangle\rangle = \langle\langle b\rangle\rangle$ then $a = b$.*
(4) *If $a + c = b + c$, then $a + u = b + u$.*
(5) *$\langle\langle a\rangle\rangle = \langle\langle b\rangle\rangle \iff a + u = b + u$.*

Item 5 motivates the following definition:

DEFINITION 2.4. Let u be an element of a monoid M.

(1) Define a congruence $\sim_u$ on M by
$$a \sim_u b \iff u + a = u + b$$
for $a, b \in M$. We will write $[a]_u$ for the $\sim_u$-congruence class containing $a \in M$ and H_u for the quotient monoid: $H_u = M/{\sim_u}$.
(2) Define $G_u = \{[a]_u \in H_u \mid a \ll u\}$. One can easily show that G_u is the set of all units (invertible elements) of H_u and so is an Abelian group.

The following facts about H_u and G_u are easy to check:

LEMMA 2.5. *Let u, v be elements of a monoid M.*

(1) *If $v \equiv u$, then $\sim_v$ and $\sim_u$ coincide. In particular, $H_u = H_v$ and $G_u = G_v$.*
(2) *The map $\Omega\colon G_u \to \{\equiv u\}$ defined by $\Omega([x]_u) = u + x$ is a bijection (but not a homomorphism). Define the operation $\square_u$ on $\{\equiv u\}$ by $a \mathbin{\square_u} b = u + x + y$ where $a = u + x$ and $b = u + y$. Then the set $\{\equiv u\}$ with operation $\square_u$ is a group isomorphic to G_u, with identity u.*

If M is strongly separative, we can strengthen these properties:

3. *If $v \propto u \propto v$, then $\sim_v$ and $\sim_u$ coincide, $H_u = H_v$ and $G_u = G_v$. In particular, this applies if u and v are order units.*
4. *The operation $\square_u$ can be expressed in a simpler way: $a \mathbin{\square_u} b = c$ where $a + b = u + c$.*

The advantage of thinking of G_u as in 2 is that the elements of the group are elements of M, rather than congruence classes. The disadvantage is that if $v \equiv u$, then $G_u = G_v$, and $\{\equiv u\} = \{\equiv v\}$, but the operations $\square_u$ and $\square_v$ are, in general, different.

We now make the connection between H_u and $G^+(M)$ explicit. Using 2.3(5) we have

LEMMA 2.6. *If u is an order unit in a strongly separative monoid M, then $G^+(M) \cong H_u$.*

3. Prime Elements in Strongly Separative Monoids

An element p of a monoid M is **prime** if for all $a_1, a_2 \in M$, $p \leq a_1 + a_2$ implies $p \leq a_1$ or $p \leq a_2$. Notice that any element $p \leq 0$ is prime. An element $p \in M$ is **proper** if $p \not\leq 0$.

We will see in 6.2 that, for a left Noetherian ring R, the monoid $M(R\text{-}\mathbf{Noeth})$ contains a proper prime element corresponding to each prime ideal of the ring. For our investigation of $G_0(R)$ it matters that there is a sum of such primes which is an order unit of $M(R\text{-}\mathbf{Noeth})$. In this section we discuss the monoid theoretic consequences of this situation in a strongly separative monoid.

First we generalize the primeness property:

Lemma 3.1. *Let p be a prime element in a strongly separative monoid M and $a_1, a_2 \in M$, $n \in \mathbb{Z}^+$ such that $np \leq a_1 + a_2$. Then there are $n_1, n_2 \in \mathbb{Z}^+$ such that $n = n_1 + n_2$, $n_1 p \leq a_1$ and $n_2 p \leq a_2$.*

Proof. The $n = 0$ case is trivial. The other cases we will prove by induction on n.

Suppose that the lemma is true for some $n \in \mathbb{Z}^+$ and there are $a_1, a_2 \in M$ such that $(n+1)p \leq a_1 + a_2$. Then, in particular, $np \leq a_1 + a_2$ and by the induction hypothesis, there are $n_1, n_2 \in \mathbb{Z}^+$ such that $n = n_1 + n_2$, $n_1 p \leq a_1$ and $n_2 p \leq a_2$. Thus $a_1 = n_1 p + b_1$ and $a_2 = n_2 p + b_2$ for some $b_1, b_2 \in M$. We now have

$$p + np \leq a_1 + a_2 = n_1 p + b_1 + n_2 p + b_2 = b_1 + b_2 + np.$$

Since M is strongly separative, we can use 2.2(3), to cancel np from this inequality to get $p \leq b_1 + b_2$. Because p is prime, we have either $p \leq b_1$ or $p \leq b_2$.

Without loss of generality we can assume $p \leq b_1$, in which case, $(n_1 + 1)p \leq a_1$ and $n_2 p \leq a_2$ with $(n_1 + 1) + n_2 = n + 1$ as required. Thus we have shown that the lemma is true for $n + 1$. □

Definition 3.2. Let p be an element in a monoid M. For $a \in M$, define $N_p(a) \in \mathbb{Z}^+ \cup \{\infty\}$ by

$$N_p(a) = \sup\{n \in \mathbb{Z}^+ \mid np \leq a\}.$$

Among the simpler properties of N_p are the following:

- $N_p(a) = 0$ if and only if $p \not\leq a$.
- If $a \leq b$, then $N_p(a) \leq N_p(b)$.
- If $a \equiv b$, then $N_p(a) = N_p(b)$.
- If $q \equiv p$, then $N_p(a) = N_q(a)$.

We can consider N_p to be a map from the monoid M to the monoid $\mathbb{Z}^+ \cup \{\infty\}$ where $n + \infty = \infty$ for all $n \in \mathbb{Z}^+$ and $\infty + \infty = \infty$. When we do so, we find that N_p is a monoid homomorphism when p is a proper prime:

Theorem 3.3. *If M is a strongly separative monoid and $p \in M$ is a proper prime element, then N_p is a monoid homomorphism and $N_p(p) = 1$.*

Proof. We show first that $N_p(a_1 + a_2) = N_p(a_1) + N_p(a_2)$ for all $a_1, a_2 \in M$.

If $n_1, n_2 \in \mathbb{Z}^+$ such that $n_1 p \leq a_1$ and $n_2 p \leq a_2$, then $(n_1 + n_2)p \leq a_1 + a_2$, so $n_1 + n_2 \leq N_p(a_1 + a_2)$. Taking the supremum over all such n_1 and n_2 gives $N_p(a_1) + N_p(a_2) \leq N_p(a_1 + a_2)$.

To show the opposite inequality, suppose $np \leq a_1 + a_2$ for some $n \in \mathbb{Z}^+$. Then from 3.1, there are $n_1, n_2 \in \mathbb{Z}^+$ such that $n = n_1 + n_2$, $n_1 p \leq a_1$ and $n_2 p \leq a_2$.

Since $n_1 \leq N_p(a_1)$ and $n_2 \leq N_p(a_2)$ we have $n \leq N_p(a_1) + N_p(a_2)$. Taking the supremum over all such n we get $N_p(a_1 + a_2) \leq N_p(a_1) + N_p(a_2)$.

Since p is proper, we also have $N_p(0) = 0$, and so N_p is a monoid homomorphism.

Finally we check that $N_p(p) = 1$. The inequality $p \leq p$ implies $N_p(p) \geq 1$. But if $N_p(p) > 1$, then we would have $2p \leq p$. We could then cancel p using 2.2(3) to get $p \leq 0$, contrary to the hypothesis. □

In the remainder of this section we will investigate the structure of a monoid M which contains prime elements $\{p_1, p_2, \ldots, p_n\}$ such that $u = p_1 + p_2 + \ldots + p_n$ is an order unit.

If we had $p_i \leq p_j$ for some $i \neq j$, then it is easy to confirm that we could remove p_i from the above sum and still have an order unit. Thus, without loss of generality, we will assume that the set $\{p_1, p_2, \ldots, p_n\}$ is **incomparable**, meaning that $p_i \leq p_j$ implies $i = j$ for $i, j \in \{1, 2, \ldots, n\}$.

We will further specialize to the case when M is not a group, meaning that M contains proper elements. In this situation, the set $\{p_1, p_2, \ldots, p_n\}$ must contain at least one proper prime and then incomparability ensures that all the remaining primes are also proper.

Now suppose that M is strongly separative. We will write N_i rather than N_{p_i} for the homomorphism corresponding to the prime element p_i. Since the set $\{p_1, p_2, \ldots, p_n\}$ is incomparable, we have

$$N_i(p_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for all $i, j \in \{1, 2, \ldots, n\}$. Also $N_i(u) = 1$ for $i \in \{1, 2, \ldots, n\}$.

For any $a \in M$ we have $a \propto u$ and so there is $m \in \mathbb{N}$ such that $a \leq mu$. Applying the homomorphism N_i we get $N_i(a) \leq N_i(mu) = m$. We have shown therefore that $N_i(a)$ is finite for $i \in \{1, 2, \ldots, n\}$.

It is convenient to combine the maps $N_1, N_2, \ldots, N_n$ into a single homomorphism $\vec{N}\colon M \to (\mathbb{Z}^+)^n$ by defining

$$\vec{N}(a) = (N_1(a), N_2(a), \ldots, N_n(a))$$

for $a \in M$. Clearly $\vec{N}(u) = (1, 1, \ldots, 1)$ and $\vec{N}(p_i) = e_i$ for $i = 1, 2, \ldots, n$ where $e_1, e_2, \ldots, e_n$ are the standard basis vectors for $\mathbb{Z}^n$.

We also define

$$\bar{a} = N_1(a)p_1 + N_2(a)p_2 + \ldots + N_n(a)p_n,$$

for any $a \in M$. One readily checks that the map $a \mapsto \bar{a}$ is a monoid homomorphism such that $\bar{\bar{a}} = \bar{a}$. Note also that $\bar{u} = u$.

Theorem 3.4. *Let M be a strongly separative monoid which contains an incomparable set $\{p_1, p_2, \ldots, p_n\}$ of proper prime elements such that $u = p_1 + p_2 + \ldots + p_n$ is an order unit. Let $a, b \in M$.*

(1) $a = \bar{a} + x$ *for some* $x \ll u$
(2) $\vec{N}(a) = (0, 0, \ldots, 0) \iff a \ll u \iff [a]_u \in G_u$
(3) $\vec{N}(a) \geq (1, 1, \ldots, 1) \iff a \geq u \iff a$ *is an order unit*
(4) *If a is an order unit, then* $(\vec{N}(a) = \vec{N}(b) \iff a \equiv b)$
(5) $G^+(M) \cong H_u \cong (\mathbb{Z}^+)^n \times G_u$

(6) $G(M) \cong \mathbb{Z}^n \times G_u$

Proof.

(1) We show first that $\bar{a} \leq a$.

By the definition of N_1 we have $N_1(a)p_1 \leq a$, so there is some $a_1 \in M$ such that $N_1(a)p_1 + a_1 = a$. Applying the homomorphism N_2 to this equation gives $N_2(a_1) = N_2(a)$. Thus there is some $a_2 \in M$ such that $a_1 = N_2(a)p_2 + a_2$, that is, $N_1(a)p_1 + N_2(a)p_2 + a_2 = a$. Repeating this process in the obvious way gives the inequality $\bar{a} \leq a$.

Write $a = \bar{a} + x$ for some $x \in M$. Since $a \propto u$, there are $b \in M$ and $m \in \mathbb{N}$ such that $a + b = mu$. We have $\bar{a} + \bar{b} = m\bar{u} = mu$, and from the previous paragraph, $\bar{b} \leq b$. Thus

$$mu + x = \bar{a} + \bar{b} + x \leq a + b = mu.$$

Using 2.2(4), we can cancel $(m-1)u$ from this inequality to get $u + x \leq u$, that is $x \ll u$.

(2) If $\vec{N}(a) = (0, 0, \ldots, 0)$, then $\bar{a} = 0$ and, by 1, $a = x$ for some $x \ll u$.

Conversely, $a \ll u$ means $a + u \leq u$. Applying the homomorphism $\vec{N}$, we get $\vec{N}(a) + \vec{N}(u) \leq \vec{N}(u)$. Since $\vec{N}(u) = (1, 1, 1, \ldots, 1)$, this implies $\vec{N}(a) = (0, 0, \ldots, 0)$.

The remaining claim is direct from the definition of G_u.

(3) If $\vec{N}(a) \geq (1, 1, \ldots, 1)$, then, with 1, we have $a \geq \bar{a} \geq u$. If $a \geq u$, then it is trivial that a is an order unit. Finally, if a is an order unit, then for some $k \in \mathbb{N}$, we have $u \leq ka$. Applying the homomorphism $\vec{N}$ we get $(1, 1, \ldots, 1) = \vec{N}(u) \leq k\vec{N}(a)$. This can only be true if $\vec{N}(a) \geq (1, 1, \ldots, 1)$.

(4) If $\vec{N}(a) = \vec{N}(b)$, then $\bar{a} = \bar{b}$, and also, from 3, b is an order unit. From 1, we have $a = \bar{a} + x$ for some $x \ll u$. Since a is an order unit, the proof of 3 shows that $u \leq \bar{a}$, and hence $x \ll \bar{a}$. Thus $a = \bar{a} + x \leq \bar{a} = \bar{b} \leq b$, that is $a \leq b$. Similarly, $b \leq a$.

The converse is easy.

(5) We define a map $\psi \colon H_u \to (\mathbb{Z}^+)^n \times G_u$ as follows: For $a \in M$ we have from 1, that $a = \bar{a} + x$ for some $x \ll u$. Thus $[x]_u \in G_u$ and we can define $\psi([a]_u) = (\vec{N}(a), [x]_u)$. Using 2.3(4), it is easy to confirm that this map is well defined.

Define $\sigma \colon (\mathbb{Z}^+)^n \times G_u \to H_u$ by $\sigma(n_1, n_2, \ldots, n_n, [x]_u) = [n_1p_1 + n_2p_2 + \ldots + n_np_n + x]_u$. It is easy to check that σ is a monoid homomorphism and that σ and ψ are inverse maps. Thus ψ is a monoid isomorphism and $H_u \cong \mathbb{Z}^+ \times G_u$. From 2.6 we also have $H_u \cong G^+(M)$.

(6) $G(M) \cong G(G^+(M)) \cong G((\mathbb{Z}^+)^n \times G_u) \cong \mathbb{Z}^n \times G_u$.

□

Notice that, from 2.5(3), parts 5 and 6 of this theorem remain true if u is replaced by any other order unit of M.

Finally, we note that the map $\vec{N} \colon M \to (\mathbb{Z}^+)^n$ has its own universal property with respect to certain types of cancellative monoids.

Corollary 3.5. *Let M be as in the theorem, and N a cancellative monoid such that $a \equiv b$ implies $a = b$ for all $a, b \in N$. If $\Lambda\colon M \to N$ is a monoid homomorphism, then Λ factors uniquely through the map $\vec{N}\colon M \to (\mathbb{Z}^+)^n$.*

Proof. Using the universal property of $G^+(M) \cong H_u \cong (\mathbb{Z}^+)^n \times G_u$, there is a unique induced monoid homomorphism $\lambda\colon (\mathbb{Z}^+)^n \times G_u \to N$ such that for all $a \in M$, $\Lambda(a) = \lambda(\vec{N}(a), [x]_u)$ where $a = \bar{a} + x$. Since G_u is a group we have $(\vec{N}(a), [x]_u) \equiv (\vec{N}(a), [0]_u)$ in $(\mathbb{Z}^+)^n \times G_u$, and so $\lambda(\vec{N}(a), [x]_u) \equiv \lambda(\vec{N}(a), [0]_u)$ in N. By hypothesis, this implies $\lambda(\vec{N}(a), [x]_u) = \lambda(\vec{N}(a), [0]_u)$, that is, $\Lambda(a) = \lambda(\vec{N}(a), [0]_u)$ for all $a \in M$. If $\lambda'\colon (\mathbb{Z}^+)^n \to N$ is the restriction of λ to $(\mathbb{Z}^+)^n$, then we have $\Lambda(a) = \lambda'(\vec{N}(a))$ for all $a \in M$. □

Examples of monoids N which satisfy the hypothesis of this corollary are $(\mathbb{Z}^+)^k$ and $(\mathbb{R}^+)^k$ for $k \in \mathbb{N}$, where $\mathbb{R}^+$ is the set of nonnegative real numbers with addition as its operation.

4. Extension Properties of Module Categories

Throughout this section, R is an arbitrary ring and R-**Noeth** the category of left Noetherian R-modules. In this section we define the monoid $M(R\text{-}\mathbf{Noeth})$ and discuss its basic properties. Not appearing here is any discussion of strong separativity. For a proof that $M(R\text{-}\mathbf{Noeth})$ is strongly separative see [**2**, 5.1].

As suggested in the introduction, $M(R\text{-}\mathbf{Noeth})$ could be defined via its universal property with respect to maps on R-**Noeth** which respect short exact sequences. We will instead construct $M(R\text{-}\mathbf{Noeth})$ in terms of certain equivalence classes of modules. The universal property then appears as 4.5.

We begin by defining some concepts which will be useful for manipulating submodule series of modules.

Definition 4.1. A **partition** of $A \in R\text{-}\mathbf{Noeth}$ is a finite indexed set of modules $\mathcal{A} = (A_i)_{i \in \mathcal{I}}$ such that there is a submodules series $0 = A'_0 \leq A'_1 \leq \cdots \leq A'_n = A$ and a bijection $\sigma\colon \mathcal{I} \to \{1, 2, .., n\}$ with $A_i \cong A'_{\sigma(i)}/A'_{\sigma(i)-1}$ for all $i \in \mathcal{I}$. Two partitions $\mathcal{A} = (A_i)_{i \in \mathcal{I}}$ and $\mathcal{B} = (B_j)_{j \in \mathcal{J}}$ are **isomorphic** if there is a bijection $\sigma\colon \mathcal{I} \to \mathcal{J}$ such that $A_i \cong B_{\sigma(i)}$ for all $i \in \mathcal{I}$.

A partition $\mathcal{B} = (B_j)_{j \in \mathcal{J}}$ is a **refinement** of partition $\mathcal{A} = (A_i)_{i \in \mathcal{I}}$ if $\mathcal{J}$ can be written as the disjoint union of subsets $(\mathcal{J}_i)_{i \in \mathcal{I}}$ such that for all $i \in \mathcal{I}$, $(B_j)_{j \in \mathcal{J}_i}$ is a partition of A_i. Note that if $\mathcal{A}$ is a partition of $A \in R\text{-}\mathbf{Noeth}$, then any refinement of $\mathcal{A}$ is also a partition of A.

If $\mathcal{A}$ and $\mathcal{B}$ are partitions then we write $\mathcal{A} \cup \mathcal{B}$ for the disjoint union of the modules in each partition indexed by the disjoint union of the corresponding index sets.

One would like to define a partition of a module A to be the *set* of isomorphism classes of the factors in some submodule series of A. Unfortunately, the same isomorphism class may appear more than once, and we want to record the multiplicity of such isomorphism classes. By making partitions indexed sets, and defining isomorphism for partitions as above we allow multiple copies of a module to appear in the partition.

If $0 \to A \to B \to C \to 0$ is a short exact sequence in R-**Noeth**, then B has the partition (A, C). Further, if $\mathcal{A}$ is partition of A and $\mathcal{C}$ is a partition of C, then $\mathcal{A} \cup \mathcal{C}$ is a partition of B.

The Schreier Refinement Theorem [**3**, 3.10], when rephrased in terms of partitions, says that any two partitions of a module have isomorphic refinements. This property is exactly what is needed to show that the claims built into the following definition are true. For the details, see [**2**, Section 3].

DEFINITION 4.2. Let $A, B \in R$-**Noeth**. We write $A \sim B$ whenever A and B have isomorphic partitions. We will write $[A]$ for the $\sim$-equivalence class containing $A \in R$-**Noeth**. Note that the zero module by itself is a $\sim$-equivalence class, that is, $[0] = \{0\}$.

We write $M(R\text{-}\mathbf{Noeth})$ for $R\text{-}\mathbf{Noeth}/\sim$, the class of $\sim$-equivalence classes of R-**Noeth**, and define the operation $+$ on $M(R\text{-}\mathbf{Noeth})$ by $[A] + [B] = [A \oplus B]$ for all $A, B \in R$-**Noeth**. $(M(R\text{-}\mathbf{Noeth}), +)$ is a commutative monoid with identity $0 = [0] = \{0\}$ which is also the only nonproper element of $M(R\text{-}\mathbf{Noeth})$.

Though the operation $+$ is defined in terms of $\oplus$, it is its relationship to short exact sequences that is crucial: If $0 \to A \to B \to C \to 0$ is an exact sequence in R-**Noeth**, then B and $A \oplus C$ both have the partition (A, C), so $B \sim A \oplus C$ and $[B] = [A \oplus C] = [A] + [C]$. A simple induction shows that if $\mathcal{A} = (A_i)_{i \in \mathcal{I}}$ is a partition of A, then $[A] = \sum_{i \in \mathcal{I}} [A_i]$.

Notice that if $A, B \in R$-**Noeth** with A isomorphic to a submodule, factor module or subfactor of B, then $[A] \leq [B]$ in $M(R\text{-}\mathbf{Noeth})$. To obtain a more precise understanding of the preorder $\leq$ as it applies to $M(R\text{-}\mathbf{Noeth})$, we use the fact that if A and B have isomorphic partitions, then any refinement of the partition of B induces an isomorphic refinement of the partition of A and vice versa.

LEMMA 4.3. *Let $A, B, A_1, A_2, B_1, B_2 \in R$-**Noeth**.*

(1) $[A_1] + [A_2] = [B_1] + [B_2]$ *in $M(R\text{-}\mathbf{Noeth})$ if and only if there are isomorphic refinements of the partitions (A_1, A_2) and (B_1, B_2).*
(2) $[A] \leq [B_1] + [B_2]$ *if and only if there is a partition $\mathcal{A} = (A_i)_{i \in \mathcal{I}}$ of A, a refinement $\mathcal{B} = (B'_j)_{j \in \mathcal{J}}$ of (B_1, B_2), and an injective map $\sigma \colon \mathcal{I} \to \mathcal{J}$ such that $A_i \cong B'_{\sigma(i)}$ for all $i \in \mathcal{I}$.*
(3) $[A] \leq [B]$ *if and only if there are partitions $\mathcal{A} = (A_i)_{i \in \mathcal{I}}$ and $\mathcal{B} = (B_j)_{j \in \mathcal{J}}$ for A and B and an injective map $\sigma \colon \mathcal{I} \to \mathcal{J}$ such that $A_i \cong B_{\sigma(i)}$ for all $i \in \mathcal{I}$.*

PROOF.

(1) If $[A_1] + [A_2] = [B_1] + [B_2]$, then $A_1 \oplus A_2 \sim B_1 \oplus B_2$, and $A_1 \oplus A_2$ and $B_1 \oplus B_2$ have isomorphic partitions. Using the Schreier Refinement Theorem, we can find a refinement of the given partition of $A_1 \oplus A_2$ which is also a refinement of (A_1, A_2). This new partition of $A_1 \oplus A_2$ induces an isomorphic refinement of the given partition of $B_1 \oplus B_2$. We can then find a further refinement of the new partition of $B_1 \oplus B_2$ that is a refinement of (B_1, B_2). This partition of (B_1, B_2) induces an isomorphic refinement of $A_1 \oplus A_2$. The resulting partitions are then isomorphic refinements of the partitions (A_1, A_2) and (B_1, B_2).
The converse is trivial.
(2) Since $[A] \leq [B_1] + [B_2]$, there is some $X \in R$-**Noeth** such that $[A] + [X] = [B_1] + [B_2]$. The claim then follows from 1.
(3) Put $B_1 = B$ and $B_2 = 0$ in 2.

$\square$

Now we consider the universal property of $M(R\text{-}\mathbf{Noeth})$:

Definition 4.4. Let N be a monoid. Then a function $\Lambda\colon R\text{-}\mathbf{Noeth} \to N$ is said to **respect short exact sequences in** R-**Noeth** if $\Lambda(0) = 0$ and $\Lambda(B) = \Lambda(A)+\Lambda(C)$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence in R-**Noeth**.

The key property of such maps is that if $\mathcal{A} = (A_i)_{i\in\mathcal{I}}$ is a partition of $A \in R\text{-}\mathbf{Noeth}$, then $\Lambda(A) = \sum_{i\in\mathcal{I}} \Lambda(A_i)$. Hence if modules A and B have isomorphic partitions, then $\Lambda(A) = \Lambda(B)$.

Theorem 4.5. *Let N be a monoid and $\Lambda\colon R\text{-}\mathbf{Noeth} \to N$, a function which respects short exact sequences. Then Λ factors uniquely through $M(R\text{-}\mathbf{Noeth})$. Specifically, there exists a unique monoid homomorphism, $\lambda\colon M(R\text{-}\mathbf{Noeth}) \to N$ such that the following diagram commutes:*

$$\begin{tikzcd} R\text{-}\mathbf{Noeth} \arrow[r, "{[\ ]}"] \arrow[dr, "\Lambda"'] & M(R\text{-}\mathbf{Noeth}) \arrow[d, "\lambda"] \\ & N \end{tikzcd}$$

Proof. Define the map $\lambda\colon M(R\text{-}\mathbf{Noeth}) \to \Lambda(R\text{-}\mathbf{Noeth})$ by $\lambda([A]) = \Lambda(A)$ for all $A \in R\text{-}\mathbf{Noeth}$. This is well defined because if $[A] = [B]$, then A and B have isomorphic partitions and so $\Lambda(A) = \Lambda(B)$. For any $[A], [B] \in M$, we have $\lambda([A] + [B]) = \lambda([A \oplus B]) = \Lambda(A \oplus B) = \Lambda(A) + \Lambda(B) = \lambda([A]) + \lambda([B])$. Also, $\lambda([0]) = \Lambda(0) = 0$. So λ is a monoid homomorphism. $\square$

Next we consider the relationship between the monoids $M(R\text{-}\mathbf{Noeth})$ and $M(R/I\text{-}\mathbf{Noeth})$ when I is a two sided ideal in R.

There is a well known functor $F\colon R/I\text{-}\mathbf{Noeth} \to R\text{-}\mathbf{Noeth}$ that takes a module ${}_{R/I}A \in R/I\text{-}\mathbf{Noeth}$ and maps it to $F(A) = {}_RA$ which has the same elements and addition as A, but with module multiplication defined by $ra = (r+I)a$ for $r \in R$ and $a \in A$. A module $A \in R\text{-}\mathbf{Noeth}$ is in $F(R/I\text{-}\mathbf{Noeth})$ if and only if it is annihilated by I. In particular, if A is in $F(R/I\text{-}\mathbf{Noeth})$, then so are any submodules, factor modules, subfactor modules, and the modules in any partition of A.

The functor F is exact, so the map $R/I\text{-}\mathbf{Noeth} \xrightarrow{F} R\text{-}\mathbf{Noeth} \xrightarrow{[\ ]} M(R\text{-}\mathbf{Noeth})$ respects short exact sequences, and there is an induced monoid homomorphism $\gamma\colon M(R/I\text{-}\mathbf{Noeth}) \to M(R\text{-}\mathbf{Noeth})$ which takes $[{}_{R/I}A]$ in $M(R/I\text{-}\mathbf{Noeth})$ to $[{}_RA]$ in $M(R\text{-}\mathbf{Noeth})$.

Theorem 4.6. *Let I be a two sided ideal in a ring R.*

1. *$M(R/I\text{-}\mathbf{Noeth})$ embeds in $M(R\text{-}\mathbf{Noeth})$ via the monoid homomorphism γ defined above.*
2. *If $[B] \le [A] \in \gamma(R/I\text{-}\mathbf{Noeth})$, then $[B] \in \gamma(R/I\text{-}\mathbf{Noeth})$.*
3. *If, in addition, $I^k = 0$ for some $k \in \mathbb{N}$, then γ is surjective. In particular, $M(R/I\text{-}\mathbf{Noeth}) \cong M(R\text{-}\mathbf{Noeth})$. If Λ is a map on R/I-**Noeth** which respects short exact sequences, then Λ extends to a unique such map on R-**Noeth** given by the formula*

$$\Lambda(A) = \sum_{j=0}^{k-1} \Lambda(I^jA/I^{j+1}A)$$

*for all $A \in R$-**Noeth**.*

PROOF.

(1) Suppose we have $A, B \in R/I$-**Noeth** such that $\gamma([A]) = \gamma([B])$, that is, $[{}_RA] = [{}_RB]$. Then ${}_RA = F(A)$ and ${}_RB = F(B)$ have isomorphic R-module partitions. From the above discussion these partitions serve also as isomorphic R/I-module partitions, that is, $[{}_{R/I}A] = [{}_{R/I}B]$ in $M(R/I\text{-}\mathbf{Noeth})$. Thus the map γ is injective and $M(R/I\text{-}\mathbf{Noeth})$ embeds in $M(R\text{-}\mathbf{Noeth})$.

(2) From 4.3(3) and the fact that all modules in any partition of a module A in $F(R/I\text{-}\mathbf{Noeth})$ are also in $F(R/I\text{-}\mathbf{Noeth})$.

(3) Any module ${}_RA \in R$-**Noeth** has the partition $(A/IA, IA/I^2A, \ldots, I^{k-1}A)$. Each module in this partition is in $F(R/I\text{-}\mathbf{Noeth})$ and so

$$[A] = [A/IA] + [IA/I^2A] + [I^2A/I^3A] + \ldots + [I^{k-1}A]$$

is in $\gamma(M(R/I\text{-}\mathbf{Noeth}))$. Consequently $[A] \in \gamma(M(R/I\text{-}\mathbf{Noeth}))$, and we have shown that γ is surjective.

The remaining claims are immediate.

□

In view of this theorem we make the following convention:

NOTATION 4.7. If I is a two sided ideal in a ring R, then we will consider $M(R/I\text{-}\mathbf{Noeth})$ to be a submonoid of $M(R\text{-}\mathbf{Noeth})$. In particular, when I is nilpotent, we have $M(R/I\text{-}\mathbf{Noeth}) = M(R\text{-}\mathbf{Noeth})$ and $G(M(R/I\text{-}\mathbf{Noeth})) = G(M(R\text{-}\mathbf{Noeth}))$.

Note that from 4.6(2), if $b \le a \in M(R/I\text{-}\mathbf{Noeth})$, then $b \in M(R/I\text{-}\mathbf{Noeth})$. This is a property not shared by all monoid embeddings. Consider, for example, the submonoid $\{0, 2, 4, 6, 8, \ldots\}$ of $\mathbb{Z}^+$.

At least one part of the monoid $M(R\text{-}\mathbf{Noeth})$ can be described explicitly, namely, the image of the finite length modules in $M(R\text{-}\mathbf{Noeth})$. Let R-**Len** be the category of all finite length modules over the ring R and $M(R\text{-}\mathbf{Len})$, the image of R-**Len** in $M(R\text{-}\mathbf{Noeth})$.

Let S be a simple left R-module. Using 4.3(2), we see that $[S]$ is a prime element of $M(R\text{-}\mathbf{Noeth})$ (and of $M(R\text{-}\mathbf{Len})$). We will write N_S rather than $N_{[S]}$ for the monoid homomorphism provided by 3.3. If $A \in R$-**Len**, then $N_S([A])$ is the number of times the isomorphism class of S occurs in a composition series for A.

Let $\mathbb{S}$ be a set of representatives of the isomorphism classes of the simple R-modules. Using the monoid homomorphisms N_S for $S \in \mathbb{S}$, it is easy to see that for any $A \in R$-**Len**,

$$[A] = \sum_{S \in \mathbb{S}} N_S([A])[S].$$

The sum makes sense since $N_S([A])$ will be finite for all $S \in \mathbb{S}$ and zero for all but a finite number of $S \in \mathbb{S}$. Further, such expressions are unique. If we write $(\mathbb{Z}^+)^{(\mathbb{S})}$ for the free monoid generated by the elements of $\mathbb{S}$, and construct the homomorphism $\nu\colon [A] \mapsto (N_S([A]))_{S \in \mathbb{S}} \in (\mathbb{Z}^+)^{(\mathbb{S})}$, then we have the following:

THEOREM 4.8. *Let R be a ring and $\mathbb{S}$ a set of representatives of the isomorphism classes of simple R-modules. Then $M(R\text{-}\mathbf{Len})$ is isomorphic to $(\mathbb{Z}^+)^{(\mathbb{S})}$ via the homomorphism ν.*

5. $G_0(R)$ and the Reduced Rank Function

As already described in the introduction, given a left Noetherian ring R, the **Grothendieck group** $G_0(R)$, is defined to be the Abelian group generated by the symbols $\langle A\rangle$ for all $A \in R$-**Noeth**, subject to the relations $\langle B\rangle = \langle A\rangle + \langle C\rangle$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence in R-**Noeth**. This group has the universal property that, if Λ is a map from R-**Noeth** into an Abelian group which respects short exact sequences, then Λ factors through $G_0(R)$. Every element of $G_0(R)$ can be written as $\langle A\rangle - \langle B\rangle$ for some $A, B \in R$-**Noeth**. We define also $G_0^+(R) = \{\langle A\rangle \mid A \in R\text{-}\mathbf{Noeth}\}$. For the basic properties of $G_0(R)$ and $G_0^+(R)$ see [**8**].

The universal properties of $M(R\text{-}\mathbf{Noeth})$, $G_0(R)$ and $G(M(R\text{-}\mathbf{Noeth}))$ imply that $G(M(R\text{-}\mathbf{Noeth}))$ and $G_0(R)$ are isomorphic via the map $\langle\langle [A]\rangle\rangle \mapsto \langle A\rangle$ for $A \in R$-**Noeth**. This same map restricts to a monoid isomorphism between $G^+(M(R\text{-}\mathbf{Noeth}))$ and $G_0^+(R)$. Accordingly, we make the following convention:

Notation 5.1. If R is a left Noetherian ring, then we will identify $G_0(R)$ and $G(M(R\text{-}\mathbf{Noeth}))$ retaining the notation used for $G_0(R)$. Similarly, we identify $G_0^+(R)$ and $G^+(M(R\text{-}\mathbf{Noeth}))$.

The reduced rank function $\rho\colon R\text{-}\mathbf{Noeth} \to \mathbb{Z}^+$ is a well known map which respects short exact sequences in R-**Noeth**. See [**3**, Ch. 10] or [**10**, 3.4.5] for its definition and properties. In this section we will redefine the reduced rank and related functions in a fashion motivated by our monoid theoretic approach.

We note first that the largest nilpotent two sided ideal of R is the prime radical. Hence we have the following corollary of 4.6 (using the conventions of 4.7 and 5.1):

Corollary 5.2. *Let R be a left Noetherian ring with prime radical N. Then $M(R/N\text{-}\mathbf{Noeth}) = M(R\text{-}\mathbf{Noeth})$, and $G_0(R/N) = G_0(R)$.*

Further, any map on R-**Noeth** which respects short exact sequences is an extension of a map defined on R/N-**Noeth**. Since R/N is a semiprime ring, we have reduced our task to defining reduced rank for semiprime rings.

Suppose then that R is a semiprime, left Noetherian ring with Goldie quotient ring Q. As a right R-module, Q is flat, so, using the universal property of $M(R\text{-}\mathbf{Noeth})$, one easily shows that there is an induced monoid homomorphism $\sigma\colon M(R\text{-}\mathbf{Noeth}) \to M(Q\text{-}\mathbf{Noeth})$ such that $\sigma([A]) = [Q \underset{R}{\otimes} A]$ for all $A \in R$-**Noeth**.

The ring Q is semisimple, so Q-**Len** $= Q$-**Noeth**, and 4.8 provides a description of $M(Q\text{-}\mathbf{Noeth})$. In this particular case, there are only a finite number of isomorphism classes of simple Q-modules. Explicitly, let $P_1, P_2, \ldots, P_n$ be the minimal prime ideals of R, and, for $i = 1, 2, \ldots, n$, let Q_i be the Goldie quotient ring of R/P_i with S_i a simple Q_i-module. Then $Q \cong Q_1 \times Q_2 \times \ldots \times Q_n$ and $\mathbb{S} = \{S_1, S_2, \ldots, S_n\}$ is a set of representatives of all the isomorphism classes of simple Q-modules. By 4.8, $M(Q\text{-}\mathbf{Noeth})$ is isomorphic to the monoid $(\mathbb{Z}^+)^n$ via the map $\vec{N}\colon M(Q\text{-}\mathbf{Noeth}) \to (\mathbb{Z}^+)^n$ given by

$$\begin{aligned}\vec{N}([A]) &= (N_{S_1}([A]), N_{S_1}([A]), \ldots, N_{S_n}([A]))\\ &= (\operatorname{len}(Q_1 \underset{Q}{\otimes} A), \operatorname{len}(Q_2 \underset{Q}{\otimes} A), \ldots, \operatorname{len}(Q_n \underset{Q}{\otimes} A))\end{aligned}$$

for $A \in Q$-**Noeth**. The homomorphism $\vec{N}$ is the same one discussed in 3.4 since $\{[S_1], [S_2], \ldots, [S_n]\}$ is an incomparable set of prime elements of $M(Q\text{-}\mathbf{Noeth})$ such that $[S_1] + [S_2] + \ldots + [S_n]$ is an order unit.

Let $\lambda\colon (\mathbb{Z}^+)^n \to \mathbb{Z}^+$ be the monoid homomorphism given by $\lambda(k_1, k_2, \ldots, k_n) = k_1 + k_2 + \ldots + k_n$. One recognizes immediately that the map from Q-**Noeth** via $M(Q\text{-}\mathbf{Noeth})$ and $(\mathbb{Z}^+)^n$ to $\mathbb{Z}^+$ takes a module $A \in Q$-**Len** and gives its composition series length $\mathrm{len}(A)$.

Summarizing this discussion in a commutative diagram we have

$$\begin{tikzcd}
R\text{-}\mathbf{Noeth} \arrow[r, "{[\ ]}"] \arrow[d, "Q\underset{R}{\otimes}-"'] \arrow[drr, bend left, "\vec{\rho}"] \arrow[drrr, bend left=50, "\rho"] & M(R\text{-}\mathbf{Noeth}) \arrow[d, "\sigma"] & & \\
Q\text{-}\mathbf{Noeth} \arrow[r, "{[\ ]}"] \arrow[rrr, bend right, "\mathrm{len}"'] & M(Q\text{-}\mathbf{Noeth}) \arrow[r, "\vec{N}"] & (\mathbb{Z}^+)^n \arrow[r, "\lambda"] & \mathbb{Z}^+
\end{tikzcd}$$

The maps $\rho\colon R\text{-}\mathbf{Noeth} \to \mathbb{Z}^+$ and $\vec{\rho}\colon R\text{-}\mathbf{Noeth} \to (\mathbb{Z}^+)^n$ are defined by this diagram. Explicitly, we have $\rho(A) = \mathrm{len}(Q \underset{R}{\otimes} A)$ and

$$\vec{\rho}(A) = (\mathrm{len}(Q_1 \underset{R}{\otimes} A), \mathrm{len}(Q_2 \underset{R}{\otimes} A), \ldots, \mathrm{len}(Q_n \underset{R}{\otimes} A))$$

for $A \in R$-**Noeth**. We will write ρ_i for the ith component of the map $\vec{\rho}$.

Using 4.6(3) we can now extend the definitions of ρ and $\vec{\rho}$ to arbitrary left Noetherian rings: Let R be a left Noetherian ring with prime radical N such that $N^k = 0$. The ring R/N is semiprime, so we have maps $\rho\colon R/N\text{-}\mathbf{Noeth} \to \mathbb{Z}^+$ and $\vec{\rho}\colon R/N\text{-}\mathbf{Noeth} \to (\mathbb{Z}^+)^n$ which respect short exact sequences. These maps extend uniquely to R-**Noeth** by the formulas

$$\rho(A) = \sum_{j=0}^{k-1} \mathrm{len}(Q \underset{R/N}{\otimes} (N^j A / N^{j+1} A))$$

and

$$\rho_i(A) = \sum_{j=0}^{k-1} \mathrm{len}(Q_i \underset{R/N}{\otimes} (N^j A / N^{j+1} A))$$

for $A \in R$-**Noeth**, and $i = 1, 2, \ldots, n$. Here Q is the Goldie quotient ring of R/N. The minimal prime ideals of R correspond to the minimal prime ideals of R/N. In particular, we still have $Q \cong Q_1 \times Q_2 \times \ldots \times Q_n$, where Q_i is the Goldie quotient ring of the ring R/P_i and $P_1, P_2, \ldots, P_n$ are the minimal prime ideals of R.

From the first equation, one recognizes that ρ is the **reduced rank function** as defined in the standard texts ([**3**, Ch. 10], [**10**, 3.4.5]). The maps ρ_i for $i = 1, 2, \ldots, n$, are called **atomic rank functions** by Krause [**6**].

We can now describe the relationship between the reduced rank function and the Grothendieck groups $G_0(R)$ and $G_0(Q)$. Notice first that since $M(Q\text{-}\mathbf{Noeth}) \cong (\mathbb{Z}^+)^n$, we have $G_0(Q) \cong G((\mathbb{Z}^+)^n) \cong \mathbb{Z}^n$. Using this fact together with the identification $M(R\text{-}\mathbf{Noeth}) = M(R/N\text{-}\mathbf{Noeth})$ we can construct the following commutative diagram:

$$\begin{tikzcd}
R\text{-}\mathbf{Noeth} \arrow[r] \arrow[rrr, bend left, "\vec{\rho}"] \arrow[dr, "\langle\ \rangle"'] & M(R\text{-}\mathbf{Noeth}) \arrow[r] \arrow[d, "\langle\!\langle\ \rangle\!\rangle"] & M(Q\text{-}\mathbf{Noeth}) \arrow[r] \arrow[d, "\langle\!\langle\ \rangle\!\rangle"] & (\mathbb{Z}^+)^n \arrow[d] \\
& G_0(R) \arrow[r, "\tau"] & G_0(Q) \arrow[r, "\cong"] & \mathbb{Z}^n
\end{tikzcd}$$

The homomorphism τ is defined using the universal property of $G_0(R)$. Explicitly

$$\tau(\langle A\rangle) = \sum_{j=0}^{k-1} \langle Q \underset{R/N}{\otimes} (N^j A/N^{j+1}A)\rangle$$

for all $A \in R$-**Noeth**. In the introduction we pointed out that since $G_0(Q) \cong \mathbb{Z}^n$ is projective, the homomorphism τ splits so that $G_0(R) \cong \mathbb{Z}^n \times \widetilde{G}_0(R)$ where $\widetilde{G}_0(R) = \ker \tau$. By diagram chasing one can easily show that for $A, B \in R$-**Noeth**, $\langle A\rangle - \langle B\rangle$ is in $\widetilde{G}_0(R)$ if and only if $\vec{\rho}(A) = \vec{\rho}(B)$.

In particular, if $\rho(A) = 0$, then $\vec{\rho}(A) = 0$ and $\langle A\rangle \in \widetilde{G}_0(R)$. It is one of the main results of this paper (6.6) that every element of $\widetilde{G}_0(R)$ has this form.

6. Prime Elements and Order Units in $M(R$-Noeth$)$

To apply the monoid theory of Section 3 to $M(R$-**Noeth**$)$ we need to identify a set of prime elements $\{p_1, p_2, \ldots, p_n\}$ of $M(R$-**Noeth**$)$ such that $u = p_1 + p_2 + \ldots + p_n$ is an order unit.

We have already seen that $[S]$ is a prime element of $M(R$-**Noeth**$)$ whenever $S \in R$-**Noeth** is a simple module. Unless the ring is Artinian, prime elements of this type will not be “large” enough. Instead, the prime elements we seek are the images of uniform submodules of R/P where P is a prime ideal of R. Following 4.7, we consider $M(R/P$-**Noeth**$)$ to be contained in $M(R$-**Noeth**$)$.

First we note a simple fact about $[R]$:

Lemma 6.1. *For any left Noetherian ring R, the element $[R]$ is an order unit of $M(R$-**Noeth**$)$. An element $[U] \in M(R$-**Noeth**$)$ is an order unit if and only if $[R] \propto [U]$.*

Proof. For any $A \in R$-**Noeth** there is an epimorphism $\sigma\colon R^m \to A$ for some $m \in \mathbb{N}$, and so $[A] \leq m[R]$, that is, $[A] \propto [R]$.

The second claim follows from the transitivity of $\propto$. □

Lemma 6.2. *Let P be a prime ideal in a left Noetherian ring R, U a uniform left submodule of R/P, and $A, A_1, A_2 \in R$-**Noeth**.*

(1) $[U] \leq [A_1] + [A_2] \iff A_1$ *or* A_2 *has a subfactor isomorphic to* U.
(2) $[U] \leq [A] \iff A$ *has a subfactor isomorphic to* U.
(3) $[U]$ *is a prime element of* $M(R$-**Noeth**$)$.
(4) $[U]$ *is an order unit of* $M(R/P$-**Noeth**$)$.

Proof.

(1) If $[U] \leq [A_1] + [A_2]$, then, from 4.3(2), U has a partition $\mathcal{U} = (U_i)_i \in \mathcal{I}$ such that for each $i \in \mathcal{I}$, U_i is isomorphic to either a subfactor of A_1 or a subfactor of A_2. Thus, without loss of generality, we can suppose that A_1 has a subfactor isomorphic to a nonzero submodule, V say, of U. Nonzero submodules of uniform modules are again uniform, so by [**7**, 3.3.3], V contains a submodule isomorphic to U. Hence A_1 has a subfactor isomorphic to U.

    The converse is trivial.
(2) Set $A_1 = A$ and $A_2 = 0$ in 1.
(3) If $[U] \leq [A_1] + [A_2]$, then, from 1, U is isomorphic to a subfactor of A_1 or A_2. Hence $[U] \leq [A_1]$ or $[U] \leq [A_2]$.
(4) From [**3**, 6.25], R/P is isomorphic to a submodule of U^m where m is the uniform dimension of R/P. Thus $[R/P] \propto [U]$. Since U is an R/P-module, the claim follows from 6.1 as applied to the ring R/P.

□

In the next lemma we use the general fact that if $A \in R$-**Noeth** is isomorphic to a submodule, factor module or subfactor of $B \in R$-**Noeth**, then the annihilator of B is contained in the annihilator of A.

LEMMA 6.3. *Let P_1 and P_2 be prime ideals in a left Noetherian ring R, and U_1, U_2 uniform left submodules of $R/P_1, R/P_2$ respectively. Then*

$$[U_1] \leq [U_2] \iff P_1 \supseteq P_2.$$

PROOF. If $[U_1] \leq [U_2]$, then from 6.2(2), U_2 has a subfactor isomorphic to U_1. Hence $P_2 = \operatorname{ann} U_2 \subseteq \operatorname{ann} U_1 = P_1$.

Conversely, if $P_1 \supseteq P_2$, then, using the fact that R/P_1 is a factor of R/P_2 and 6.2(4), we get $[U_1] \leq [R/P_1] \leq [R/P_2] \propto [U_2]$. Thus $[U_1] \propto [U_2]$. Since $[U_1]$ is prime, this implies $[U_1] \leq [U_2]$. □

Throughout the remainder of this section $P_1, P_2, \dots, P_n$ will be the minimal prime ideals of a left Noetherian ring R. Using 4.7, we will consider the monoids $M(R/P_i$-**Noeth**$)$ for $i \in \{1, 2, \dots, n\}$ to be contained in $M(R$-**Noeth**$)$. For each $i \in \{1, 2, \dots, n\}$, let U_i be a uniform submodule of R/P_i, and $p_i = [U_i] \in M(R$-**Noeth**$)$.

THEOREM 6.4. *Let R be a left Noetherian ring, and, for $i \in \{1, 2, \dots, n\}$, let P_i, U_i, p_i be as above. Then $\{p_1, p_2, \dots, p_n\}$ is an incomparable set of proper prime elements such that $u = p_1 + p_2 + \ldots + p_n$ is an order unit in $M(R$-**Noeth**$)$.*

PROOF. From 6.2(3) and 6.3, $\{p_1, p_2, \dots, p_n\}$ is an incomparable set of prime elements of $M(R$-**Noeth**$)$, and since these primes are nonzero, they are proper.

It remains then only to prove that u is an order unit in $M(R$-**Noeth**$)$.

From [**3**, 2.4] there is a finite product of the minimal prime ideals $P_1, P_2, \dots, P_n$ (repetitions allowed) which equals zero. Let $P'_k \dots P'_2 P'_1 = 0$ with $P'_1, P'_2, \dots, P'_k \in \{P_1, P_2, \dots, P_n\}$ be such a product.

Given $A \in R$-**Noeth**, we have the submodule series $A \geq P'_1 A \geq P'_2 P'_1 A \geq \ldots \geq 0$. The ith factor in the series is in R/P'_i-**Noeth** and so the corresponding term in the expression

$$[A] = [A/P'_1 A] + [P'_1 A / P'_2 P'_1 A] + \ldots + [P'_{k-1} \dots P'_2 P'_1 A]$$

is in $M(R/P'_i$-**Noeth**$)$. Thus $[A] \in M(R/P_1$-**Noeth**$) + M(R/P_2$-**Noeth**$) + \ldots + M(R/P_n$-**Noeth**$)$.

We have shown then that

$$M(R\text{-}\mathbf{Noeth}) = M(R/P_1\text{-}\mathbf{Noeth}) + M(R/P_2\text{-}\mathbf{Noeth}) + \ldots + M(R/P_n\text{-}\mathbf{Noeth}).$$

Since by 6.2(4), $p_i = [U_i]$ is an order unit of $M(R/P_i\text{-}\mathbf{Noeth})$ for $i = 1, 2, \ldots, n$, it follows easily that $u = p_1 + p_2 + \ldots + p_n$ is an order unit of $M(R\text{-}\mathbf{Noeth})$. □

Since $M(R\text{-}\mathbf{Noeth})$ is strongly separative [**2**, 5.1], we can apply the monoid structure theory in Section 3. In particular there is a monoid homomorphism $\vec{N}\colon M(R\text{-}\mathbf{Noeth}) \to (\mathbb{Z}^+)^n$ defined with respect to the primes $\{p_1, p_2, \ldots, p_n\}$. We make the connection between this monoid theoretic rank function and the reduced rank function $\vec{\rho}$:

THEOREM 6.5. *Let R be a left Noetherian ring, and $\vec{N}\colon M(R\text{-}\mathbf{Noeth}) \to (\mathbb{Z}^+)^n$ be as described above. Then $\vec{N}([A]) = \vec{\rho}(A)$ for all $A \in R\text{-}\mathbf{Noeth}$.*

PROOF. The map $\vec{\rho}\colon R\text{-}\mathbf{Noeth} \to (\mathbb{Z}^+)^n$ respects short exact sequences, so using the universal property of $M(R\text{-}\mathbf{Noeth})$ and then 3.5, there is a unique monoid homomorphism $\mu\colon (\mathbb{Z}^+)^n \to (\mathbb{Z}^+)^n$ such that $\vec{\rho}(A) = \mu(\vec{N}([A]))$ for all $A \in R\text{-}\mathbf{Noeth}$.

To prove that μ is the identity function, it suffices to show that $\mu(e_i) = e_i$ for the standard basis vectors $e_1, e_2, \ldots, e_n$ of $(\mathbb{Z}^+)^n$. For $i = 1, 2, \ldots, n$ we have $\vec{N}(p_i) = \vec{N}([U_i]) = e_i$, and it is easy to confirm that $\vec{\rho}(U_i) = e_i$ [**6**, 2.3], and so $\mu(e_i) = e_i$ as required. □

THEOREM 6.6. *Let R be a left Noetherian ring, $P_1, P_2, \ldots, P_n$ the minimal prime ideals of R. For $i = 1, 2, \ldots, n$ let U_i be a uniform submodule of R/P_i and $p_i = [U_i]$. Let $u = p_1 + p_2 + \ldots + p_n$. Let $A, B \in R\text{-}\mathbf{Noeth}$.*

(1) $\rho(A) = 0 \iff \vec{\rho}(A) = 0 \iff [A] \ll u \iff [[A]]_u \in G_u$
(2) $\vec{\rho}(A) \geq (1, 1, \ldots, 1) \iff [A]$ *is an order unit*
(3) *If* $[A]$ *is an order unit, then* $(\vec{\rho}(A) = \vec{\rho}(A) \iff [A] \equiv [B])$
(4) $G_0^+(R) \cong H_u \cong (\mathbb{Z}^+)^n \times G_u$
(5) $G_0(R) \cong \mathbb{Z}^n \times G_u$
(6) $G_u \cong \widetilde{G}_0(R) = \{\langle A \rangle \mid A \in R\text{-}\mathbf{Noeth} \text{ and } \rho(A) = 0\}$

PROOF. Items 1-5 are directly from 3.4 and 6.5.

Since we now have $G_0(R) \cong \mathbb{Z}^n \times G_u$ and $G_0(R) \cong \mathbb{Z}^n \times \widetilde{G}_0(R)$, it is no surprise that $G_u \cong \widetilde{G}_0(R)$. Confirming this takes a little effort:

Tracing through the maps implicit in 3.4, and using 6.5, one finds that the isomorphism $G_0^+(R) \cong (\mathbb{Z}^+)^n \times G_u$ is given by the map $\langle A \rangle \mapsto (\vec{\rho}(A), [x]_u)$ where $x \in M(R\text{-}\mathbf{Noeth})$ is determined by the equation $[A] = \overline{[A]} + x$. Hence the isomorphism $G_0(R) \cong \mathbb{Z}^n \times G_u$ is given by the map $\langle A \rangle - \langle B \rangle \mapsto (\vec{\rho}(A) - \vec{\rho}(B), [x]_u - [y]_u)$ for suitable $x, y \in M(R\text{-}\mathbf{Noeth})$. Since $\langle A \rangle - \langle B \rangle \in \widetilde{G}_0(R)$ if and only if $\vec{\rho}(A) = \vec{\rho}(B)$, it is easy to see that under this isomorphism, $\widetilde{G}_0(R)$ maps onto $0 \times G_u \subseteq \mathbb{Z}^n \times G_u$. Therefore G_u and $\widetilde{G}_0(R)$ are isomorphic via the map $[[A]]_u \mapsto \langle A \rangle$ for all $[[A]]_u \in G_u$. From 1 we see that every element of $\widetilde{G}_0(R)$ is then of the form $\langle A \rangle$ for some $A \in R\text{-}\mathbf{Noeth}$ such that $\rho(A) = 0$. □

Using 2.5(3) one easily confirms that all the claims of this theorem remain true if u is replaced by any other order unit of $M(R\text{-}\mathbf{Noeth})$. For example, as described in the introduction, we have $G_0^+(R) \cong (\mathbb{Z}^+)^n \times G_r$, $G_0(R) \cong \mathbb{Z}^n \times G_r$ and $\widetilde{G}_0(R) \cong G_r$ where $r = [R]$.

Perhaps the most interesting consequence of the theorem is that $\widetilde{G}_0(R)$ is embedded in $M(R\text{-}\mathbf{Noeth})$: From 2.5(2) we have that $\widetilde{G}_0(R)$ is in bijection with the set $\{\equiv u\}$, or more generally with the set $\{\equiv v\}$ for any order unit v. Moreover, by 4.6, for any two sided ideal I, $M(R/I\text{-}\mathbf{Noeth})$ embeds in $M(R\text{-}\mathbf{Noeth})$. Since $[R/I]$ is an order unit of $M(R/I\text{-}\mathbf{Noeth})$ we get the following corollary.

Corollary 6.7. *Let I be a two sided ideal in a left Noetherian ring R and $u_I = [R/I] \in M(R\text{-}\mathbf{Noeth})$. Then $\widetilde{G}_0(R/I)$ is isomorphic to the set $\{\equiv u_I\}$ with operation $\square_I$ defined by $a \square_I b = c$ where $a + b = u_I + c$ for $a, b, c \in \{\equiv u_I\}$.*

This corollary follows directly from 2.5, but requires some attention to the question of whether $\{\equiv u_I\}$ as a subset of $M(R/I\text{-}\mathbf{Noeth})$ is the same as $\{\equiv u_I\}$ as a subset of $M(R\text{-}\mathbf{Noeth})$. That no ambiguity arises follows from the comment following 4.7.

Finally in this section we record a universal property for the reduced rank function.

Corollary 6.8. *Let N be a cancellative monoid such that $a \equiv b$ implies $a = b$ for all $a, b \in N$, and $\Lambda\colon R\text{-}\mathbf{Noeth} \to N$, a map which respects short exact sequences in $R\text{-}\mathbf{Noeth}$. Then Λ factors uniquely through the map $\vec{\rho}\colon R\text{-}\mathbf{Noeth} \to (\mathbb{Z}^+)^n$.*

Proof. From 4.5, 3.5 and 6.5. $\square$

This corollary can be phrased as follows: If $\Lambda\colon R\text{-}\mathbf{Noeth} \to N$ is a map as above, then $\Lambda(A) = k_1\rho_1(A) + k_2\rho_2(A) + \ldots + k_n\rho_n(A)$ for all $A \in R\text{-}\mathbf{Noeth}$ where $k_i = \Lambda(U_i)$ for $i = 1, 2, \ldots, n$. Krause [**6**] proved the special case of this corollary when $N = \mathbb{Z}^+$.

7. Generators and Relations for $\widetilde{G}_0(R)$

Combining the results of the previous sections, we are now able to prove the main theorem of the paper which provides generators and relations for $\widetilde{G}_0$ of any left Noetherian ring.

As in the previous section, $P_1, P_2, \ldots, P_n$ are the minimal prime ideals of a left Noetherian ring R. For each $i \in \{1, 2, \ldots, n\}$, U_i is a uniform submodule of R/P_i, and $p_i = [U_i] \in M(R\text{-}\mathbf{Noeth})$. We also set $u = p_1 + p_2 + \ldots + p_n = [U_1 \oplus U_2 \oplus \ldots \oplus U_n]$, and remind the reader that, from 6.6(6), the group G_u, defined in 2.4, and $\widetilde{G}_0(R)$ are isomorphic.

We define $R\text{-}\mathbf{Tor} = \{A \in R\text{-}\mathbf{Noeth} \mid \rho(A) = 0\}$. When R is semiprime left Noetherian, the modules in $R\text{-}\mathbf{Tor}$ are exactly the R-torsion modules ([**3**, 10.5]). If R is not semiprime, then it is not clear how to define torsion modules, so, for the purposes of this paper, we are free to consider that any module with reduced rank zero is torsion. We also define $M(R\text{-}\mathbf{Tor})$ to be the image of $R\text{-}\mathbf{Tor}$ in $M(R\text{-}\mathbf{Noeth})$.

From 6.6(1) we have that for all $A \in R\text{-}\mathbf{Noeth}$,

$$A \in R\text{-}\mathbf{Tor} \iff \rho(A) = 0 \iff [A] \ll u \iff [[A]]_u \in G_u.$$

Thus the map $M(R\text{-}\mathbf{Tor}) \to G_u$ taking $[A]$ to $[[A]]_u$ is surjective, and G_u is $M(R\text{-}\mathbf{Tor})$ modulo the congruence $\sim_u$. Our goal is to provide a module theoretic description of this congruence and hence of G_u and $\widetilde{G}_0(R)$ — at least in circumstances where $M(R\text{-}\mathbf{Tor})$ is understood.

The key to this description is the following simple observation: For some $i \in \{1, 2, \ldots, n\}$, let U_i' be a submodule of U_i such that $U_i' \cong U_i$. Since $\rho(U_i) = \rho(U_i')$, we have $\rho(U_i/U_i') = 0$, that is, $U_i/U_i' \in R$-**Tor**. Further, $[U_i] = [U_i'] + [U_i/U_i'] = [U_i] + [U_i/U_i']$, and since $[U_i] \le u$, this implies that $u = u + [U_i/U_i']$. In G_u, this means that $[[U_i/U_i']]_u = [0]_u = 0$. Thus for each submodule $U_i' \le U_i$ such that $U_i' \cong U_i$, U_i/U_i' is a torsion module which becomes trivial in G_u. This suggests the following definition:

Definition 7.1. Let $\approx$ be the congruence on $M(R\text{-}\mathbf{Tor})$ which is generated by

$$[U_i/U_i'] \approx 0$$

whenever $i \in \{1, 2, \dots, n\}$ and $U_i' \le U_i$ is such that $U_i' \cong U_i$. We write $G_\approx = M(R\text{-}\mathbf{Tor})/\approx$ for the quotient monoid, and $\langle a\rangle_\approx = \langle [A]\rangle_\approx$ for the image of $a = [A] \in M(R\text{-}\mathbf{Tor})$ in $G_\approx$.

Our goal is to show that for $A, B \in R$-**Tor**, $[A] \approx [B]$ if and only if $[A] \sim_u [B]$, and hence that $G_\approx = G_u$.

We have chosen the generators of the congruence $\approx$ so that $[A] \approx [B]$ implies $[A] \sim_u [B]$. To prove the opposite implication we will need to construct a map $\Delta\colon R\text{-}\mathbf{Noeth} \to G_\approx$ which respects short exact sequences. We define this map first on certain partitions of modules:

Definition 7.2. Let $A \in R$-**Noeth**. A **tu-partition** (tu=torsion-uniform) of A is a partition $\mathcal{A} = (A_j)_{j\in\mathcal{J}}$ of A such that for each $j \in \mathcal{J}$ either $A_j \in R$-**Tor** or $A_j = U_i$ for some $i \in \{1, 2, \dots, n\}$.

For a tu-partition $\mathcal{A} = (A_j)_{j\in\mathcal{J}}$ we define $\Delta(\mathcal{A}) \in G_\approx$ by

$$\Delta(\mathcal{A}) = \sum_{A_j \in R\text{-}\mathbf{Tor}} \langle [A_j]\rangle_\approx.$$

Of course, if A is a torsion module, then any partition $\mathcal{A}$ of A is a tu-partition and $\Delta(\mathcal{A}) = \langle [A]\rangle_\approx$. In particular, $\Delta(\mathcal{A})$ is the same for all tu-partitions of A. This property we want to extend to the case that A is not torsion.

Lemma 7.3. *Let $A \in R$-**Noeth**.*

(1) *A has a tu-partition.*
(2) *Any partition of A has a refinement which is a tu-partition.*
(3) *If $\mathcal{U}$ is a tu-partition of U_i for some $i \in \{1, 2, \dots, n\}$, then $\Delta(\mathcal{U}) = 0$.*
(4) *If $\mathcal{A}$ and $\mathcal{A}'$ are tu-partitions of A with $\mathcal{A}'$ a refinement of $\mathcal{A}$, then $\Delta(\mathcal{A}') = \Delta(\mathcal{A})$.*
(5) *If $\mathcal{A}$ and $\mathcal{A}'$ are tu-partitions of A, then $\Delta(\mathcal{A}') = \Delta(\mathcal{A})$.*

Proof.

(1) An induction on the reduced rank of A...

If $\rho(A) = 0$, then $A \in R$-**Tor** and we are done.

Suppose the claim is true for modules of reduced rank $m \in \mathbb{Z}^+$ or less, and $\rho(A) = m + 1$. That $\rho(A) > 0$ means that $\rho_i(A) > 0$ for some $i \in \{1, 2, \dots, n\}$ and hence $[U_i] \le [A]$. By 6.2(2), A has submodules $A' < A''$ such that $A''/A' \cong U_i$. Since $\rho(U_i) = 1$, we have $\rho(A/A'') \le m$ and $\rho(A') \le m$. Hence A/A'' and A' have tu-partitions $\mathcal{A}'$ and $\mathcal{A}''$. Then $\mathcal{A}' \cup \mathcal{A}'' \cup \{U_i\}$ is a tu-partition of A.

(2) Immediate from 1, since any module in the partition of A has a tu-partition.

(3) Since the modules in $\mathcal{U}$ are isomorphic to the factors of a submodule series in U_i, exactly one of these modules is isomorphic to a submodule U_i' of U_i, and all others must be torsion modules. Thus, without loss of generality, we can assume that $\mathcal{U} = (U_i', A_1, A_2, \ldots, A_k)$ where $A_1, A_2, \ldots, A_k$ are torsion modules. Notice also that $[U_i/U_i'] = [A_1] + [A_2] + \ldots + [A_k]$ and $\vec{\rho}(U_i') = \vec{\rho}(U_i)$.

Since U_i' is not torsion, it must be isomorphic to one of $U_1, U_2, \ldots, U_n$. The equation $\vec{\rho}(U_i') = \vec{\rho}(U_i)$ shows that we must have $U_i' \cong U_i$.

Putting everything together we get

$$\begin{aligned}\Delta(\mathcal{U}) &= \langle [A_1] \rangle_\approx + \langle [A_2] \rangle_\approx + \ldots + \langle [A_k] \rangle_\approx \\ &= \langle ([A_1] + [A_2] + \ldots + [A_k]) \rangle_\approx \\ &= \langle [U_i/U_i'] \rangle_\approx = 0.\end{aligned}$$

(4) This follows from 3, since $\mathcal{A}'$ is a union of tu-partitions of U_i for various indexes, and tu-partitions of torsion modules.

(5) Since $\mathcal{A}$ and $\mathcal{A}'$ are partitions of A, they have a common refinement $\mathcal{A}''$. By 2, we can assume $\mathcal{A}''$ is a tu-partition. Then by 4, $\Delta(\mathcal{A}) = \Delta(\mathcal{A}'') = \Delta(\mathcal{A}')$.

$\square$

Using items 1 and 5 of this lemma we can now define the map $\Delta\colon R\text{-}\mathbf{Noeth} \to G_\approx$.

Definition 7.4. For $A \in R\text{-}\mathbf{Noeth}$, define $\Delta(A) = \Delta(\mathcal{A})$ where $\mathcal{A}$ is any tu-partition of A.

It is simple to confirm that $\Delta\colon R\text{-}\mathbf{Noeth} \to G_\approx$ respects short exact sequences: If $0 \to A \to B \to C \to 0$ is an exact sequence in R-**Noeth**, then the union of tu-partitions for A and C is a tu-partition of B, and so $\Delta(B) = \Delta(A) + \Delta(C)$. Consequently there is an induced monoid homomorphism $\delta\colon M(R\text{-}\mathbf{Noeth}) \to G_\approx$ such that $\delta([A]) = \Delta(A)$ for all $A \in R\text{-}\mathbf{Noeth}$.

Reviewing the situation for some special cases we have $\delta([A]) = \Delta(A) = \langle [A] \rangle_\approx$ for any $A \in R\text{-}\mathbf{Tor}$, $\delta(p_i) = \Delta(U_i) = 0$ for $i = 1, 2, \ldots, n$, and $\delta(u) = \delta(p_1 + p_2 + \ldots + p_n) = 0$.

The monoid homomorphism $\delta\colon M(R\text{-}\mathbf{Noeth}) \to G_\approx$ is used in an essential way to show that $G_\approx = G_u$, and hence prove the main theorem of this paper that $\widetilde{G}_0(R) \cong G_\approx$:

Theorem 7.5. *Let R be a left Noetherian ring, $P_1, P_2, \ldots, P_n$ the minimal prime ideals of R. For $i = 1, 2, \ldots, n$ let U_i be a uniform submodule of R/P_i. Then $\widetilde{G}_0(R)$ is isomorphic to the monoid $M(R\text{-}\mathbf{Tor})$ modulo the congruence $\approx$ generated by the relations*

$$[U_i/U_i'] \approx 0$$

whenever $i \in \{1, 2, \ldots, n\}$ with $U_i' \leq U_i$ is such that $U_i' \cong U_i$.

Proof. We first show that $G_\approx$ is a group. Since $G_\approx$ is a monoid, it suffices to show that every element of $G_\approx$ has an inverse.

Let $\langle [A] \rangle_\approx \in G_\approx$ for some $A \in R\text{-}\mathbf{Tor}$. Then $[A] \ll u$ so there is some $B \in R\text{-}\mathbf{Noeth}$ such that $[A] + [B] = u$. Applying the homomorphism δ we get $\delta([A]) + \delta([B]) = \delta(u)$, or $\langle [A] \rangle_\approx + \Delta(B) = 0$. Thus $\langle [A] \rangle_\approx$ has an inverse in $G_\approx$.

We have already noted that $[A] \approx [B]$ implies $[A] \sim_u [B]$ for all $A, B \in R$-**Tor**. To show the converse, we use the universal property of $G_0^+(R)$ and the fact that, by 6.6(4), $G_0^+(R)$ is isomorphic to H_u.

Since $G_\approx$ is cancellative, the monoid homomorphism $\delta\colon M(R\text{-}\mathbf{Noeth}) \to G_\approx$ induces a monoid homomorphism $\delta'\colon H_u \to G_\approx$ such that $\delta'([[A]]_u) = \delta([A]) = \Delta(A)$ for all $A \in R$-**Noeth**.

If $A, B \in R$-**Tor** such that $[A] \sim_u [B]$, then $[[A]]_u = [[B]]_u$ in H_u and so

$$\langle [A] \rangle_\approx = \Delta(A) = \delta'([[A]]_u) = \delta'([[B]]_u) = \Delta(B) = \langle [B] \rangle_\approx,$$

that is, $[A] \approx [B]$.

We have now shown that if $A, B \in R$-**Tor**, then $[A] \approx [B]$ if and only if $[A] \sim_u [B]$, and hence that $G_\approx = G_u$. That $\widetilde{G}_0(R) \cong G_\approx$ follows from 6.6(6). $\square$

For the remainder of this section we consider the case of prime rings with Krull dimension 1. In this case, 0 is the unique minimal prime ideal and we need only one uniform ideal $U \le R$ in the theorem. Since, in addition, the ring has Krull dimension 1, a module A is in R-**Tor** if and only if A has finite length ([**3**, 13.7]), that is, we have R-**Tor** $= R$-**Len**. From 4.8, R-**Tor** is the free monoid generated by the elements $\{[S] \mid S \in \mathbb{S}\}$ where $\mathbb{S}$ is a set of representatives of the isomorphism classes of simple R-modules.

Another simplification is that $\approx$ is generated by the relations $[U/U'] \approx 0$ where U' is a maximal (proper) image of U in itself. To show this, suppose we have some $U' \le U$ such that $U' \cong U$. If U' is not a maximal image of U in itself, there is some $U_1 < U$ such that $U' \le U_1$ and U_1 is a maximal image of U in U. If U' is not a maximal image of U in U_1, we repeat to produce a descending chain $U = U_0 > U_1 > U_2 > \ldots > U'$.

The quotient module U/U' has finite length, so this chain must be finite with $U_k = U'$ for some $k \in \mathbb{N}$. The relation $[U/U'] \approx 0$ is then generated as a congruence by the relations $[U_i/U_{i+1}] \approx 0$ for $i = 0, 1, 2, \ldots, k-1$. Since each of the factors in the submodule series is isomorphic to U modulo a maximal proper image of U in U, each of these generating relations are of the claimed form.

Therefore we have

Corollary 7.6. *Let R be a left Noetherian prime ring with Krull dimension 1, $U \le R$ a uniform left ideal, and $\mathbb{S}$ a set of representatives of the isomorphism classes of simple left R-modules. Then $\widetilde{G}_0(R)$ is isomorphic to the monoid generated by the symbols $\{[S] \mid S \in \mathbb{S}\}$, modulo the congruence generated by $[S_1] + [S_2] + \ldots + [S_k] \approx 0$ whenever $(S_1, S_2, \ldots, S_k)$ is a list of the composition factors of U/U' with U' maximal among (proper) submodules of U which are isomorphic to U.*

For those who prefer a group theoretic version of this theorem, some diagram chasing shows

Corollary 7.7. *Let R be a left Noetherian prime ring with Krull dimension 1, $U \le R$ a uniform left ideal, and $\mathbb{S}$ a set of representatives of the isomorphism classes of simple left R-modules. Then $\widetilde{G}_0(R)$ is isomorphic to the Abelian group with one generator $[S]$ for each $S \in \mathbb{S}$, and relations $[S_1] + [S_2] + \ldots + [S_k] = 0$ whenever $(S_1, S_2, \ldots, S_k)$ is a list of the composition factors of U/U' with U' maximal among (proper) submodules of U which are isomorphic to U.*

If R is a left Noetherian domain, then we get further simplifications: Firstly, R is itself uniform; secondly, an ideal of R is isomorphic to R if and only if it is principal; finally, a proper principal ideal Rx is maximal among proper principal ideals if and only if x is **irreducible**, meaning that x is not a unit, and for $a, b \in R$, $x = ab$ implies a is a unit or b is a unit.

COROLLARY 7.8. *Let R be a left Noetherian domain with Krull dimension 1, and $\mathbb{S}$ a set of representatives of the isomorphism classes of simple left R-modules. Then $\widetilde{G}_0(R)$ is isomorphic to the Abelian group with one generator $[S]$ for each $S \in \mathbb{S}$, and relations $[S_1] + [S_2] + \ldots + [S_k] = 0$ whenever $(S_1, S_2, \ldots, S_k)$ is a list of the composition factors of R/Rx with $x \in R$ irreducible.*

Theorem 1.1 from the introduction is this corollary rewritten with the notation used for $G_0(R)$.

Since R is prime, in any of the circumstances of the last three corollaries we have $G_0(R) \cong \mathbb{Z} \times \widetilde{G}_0(R)$.

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