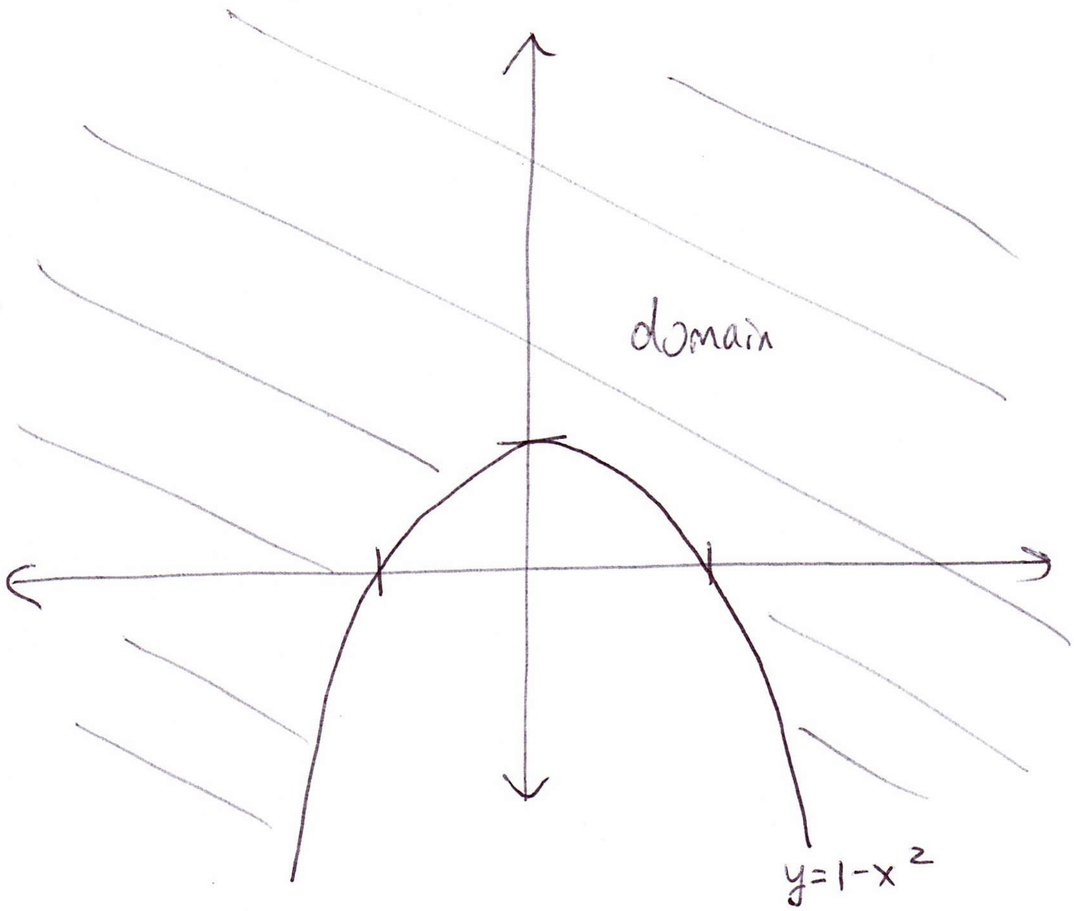
Test 1, Math 209, Fall 2015 **Name:** Solutions

Directions: Show all of your work to get credit. No calculators. Good luck!

1. [5 points] Find and sketch the domain of $f(x, y) = \sqrt{y + x^2 - 1}$

Need $y + x^2 - 1 \geq 0$ or $y \geq 1 - x^2$



2. [10 points - 5 each] Find the limit, if it exists, or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{4+yx^2}{1-2x^2} = \frac{4+(1)(1)^2}{1-2(1)^2} = \frac{4+1}{1-2} = \frac{5}{-1} = -5$

rational functions are continuous on their domains

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2ye^x}{x^3+y^3}$

Let $x=0$ and $y \neq 0$. Then $\dfrac{x^2ye^x}{x^3+y^3} = 0$.

not equal

Same for $y=0$ and $x \neq 0$.

Let $y=x$ and $x \neq 0$, $y \neq 0$. Then

$$\frac{x^2ye^x}{x^3+y^3} = \frac{x^2 \cdot xe^x}{x^3+x^3} = \frac{x^3e^x}{2x^3} = \frac{e^x}{2} \to \frac{e^0}{2} = \frac{1}{2}$$

as $x \to 0$

Since $0 \neq \frac{1}{2}$ the limit does not exist.

3. [5 points] Use the limit definition of partial derivative to find f_y where

$$f(x,y) = \frac{y}{y+x^2}$$

You must use the limit definition to get any credit.

$$\lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0} \frac{\frac{y+h}{y+h+x^2} - \frac{y}{y+x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(y+h)(y+x^2) - y(y+h+x^2)}{(y+h+x^2)(y+x^2)}}{h}$$

$$= \lim_{h\to 0} \frac{y^2+yx^2+hy+hx^2-y^2-yh-yx^2}{h(y+h+x^2)(y+x^2)}$$

$$= \lim_{h\to 0} \frac{hx^2}{h(y+h+x^2)(y+x^2)} =$$

$$= \lim_{h\to 0} \frac{x^2}{(y+h+x^2)(y+x^2)} = \frac{x^2}{(y+x^2)^2}$$

<u>Check:</u>

$$\frac{\partial}{\partial y}\left(\frac{y}{y+x^2}\right) = \frac{1\cdot(y+x^2)-y(1)}{(y+x^2)^2} = \frac{x^2}{(y+x^2)^2} \checkmark$$

4. [10 points - 5 each]

(a) Let $f(x, y) = 10x^3y^2 + \cos(2xy^3)$. Find f_x.

$$f_x = 30x^2y^2 - \sin(2xy^3) \cdot 2y^3$$

$$f_x = 30x^2y^2 - 2y^3\sin(2xy^3)$$

(b) Let $g(x, y) = \ln(10x - x^3y^2) + 10x^3$. Find g_y.

$$g_y = \frac{1}{10x - x^3y^2} \cdot (-2x^3y) = \frac{-2x^3y}{10x - x^3y^2}$$

5. [5 points] If $z = x^2\sin(y) + ye^{xy}$, where $x = s + 2t$ and $y = st$, use the chain rule to find $\frac{\partial z}{\partial s}$ where $s = 0$ and $t = 1$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial x} = 2x\sin(y) + (ye^{xy})\cdot y = 2x\sin(y) + y^2e^{xy}$$

$$\frac{\partial z}{\partial y} = x^2\cos(y) + e^{xy} + (ye^{xy})\cdot x$$

$$\frac{\partial z}{\partial y} = x^2\cos(y) + e^{xy} + xye^{xy}$$

$$\frac{\partial x}{\partial s} = 1 \quad \text{and} \quad \frac{\partial y}{\partial s} = t$$

When $s=0$ and $t=1$ we have that $x = 0+2(1) = 2$ and $y=0$. So, when $s=0$ and $t=1$ we have

$$\frac{\partial z}{\partial s} = \underbrace{\left(2(2)\sin(0) + 0^2e^0\right)}_{\frac{\partial z}{\partial x}}\cdot\underbrace{(1)}_{\frac{\partial x}{\partial s}} + \underbrace{\left(2^2\cos(0) + e^0 + 0e^0\right)}_{\frac{\partial z}{\partial y}}\underbrace{(1)}_{\frac{\partial y}{\partial s}}$$

$$\frac{\partial z}{\partial s} = 0 + (4+1+0)(1) = 5$$

6. [5 points] Find the linear approximation of $f(x, y) = \sqrt{x + xy}$ at the point $(1, 3)$. Then approximate $f(1.1, 3.1)$

$$f(x,y) = (x+xy)^{1/2}$$

$$f_x(x,y) = \frac{1}{2}(x+xy)^{-1/2} \cdot (1+y) = \frac{1+y}{2\sqrt{x+xy}}$$

$$f_y(x,y) = \frac{1}{2}(x+xy)^{-1/2} \cdot (x) = \frac{x}{2\sqrt{x+xy}}$$

$$f(1,3) = (1+3)^{1/2} = 4^{1/2} = 2$$

$$f_x(1,3) = \frac{1+3}{2\sqrt{1+3}} = \frac{4}{2\sqrt{4}} = 1$$

$$f_y(1,3) = \frac{1}{2\sqrt{1+3}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$L(x,y) = 2 + 1 \cdot (x-1) + \frac{1}{4}(y-3)$$

$$\boxed{L(x,y) = 2 + x - 1 + \frac{1}{4}y - \frac{3}{4} = \frac{1}{4} + x + \frac{1}{4}y}$$

$$f(1.1, 3.1) \approx L(1.1, 3.1) = \frac{1}{4} + 1.1 + \frac{1}{4}(3.1)$$

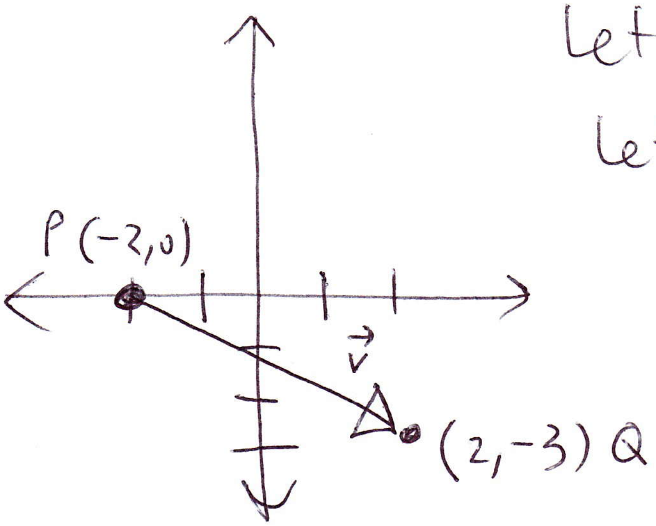
2.125

7. [5 points] Find the directional derivative of $f(x, y) = x^2 e^{-y}$ at the point $(-2, 0)$ in the direction toward the point $(2, -3)$.

$$\nabla f = \langle 2xe^{-y}, -x^2 e^{-y} \rangle$$

$$\nabla f(-2,0) = \langle 2(-2)e^0, -(-2)^2 e^0 \rangle$$

$$= \langle -4, -4 \rangle$$



Let $P = (-2,0)$ and $Q = (2,-3)$.

Let $\vec{v} = \overrightarrow{PQ} = \langle 2-(-2), -3-0 \rangle$

$= \langle 4, -3 \rangle$.

Then $\|\vec{v}\| = \sqrt{4^2 + (-3)^2}$

$= \sqrt{16+9} = 5$.

Let $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{5}\langle 4,-3 \rangle$

$= \langle \frac{4}{5}, \frac{-3}{5} \rangle$.

Then

$D_{\vec{u}} f(-2,0) = \nabla f(-2,0) \cdot \vec{u}$

$= \langle -4,-4 \rangle \cdot \langle \frac{4}{5}, -\frac{3}{5} \rangle$

$= \frac{-16}{5} + \frac{12}{5} = \frac{-4}{5}$