Test 2, Math 4650, Fall 2017 **Name:** Solution

1. [10 points] Use the definition of limit to show that

$$f(x) = \sqrt{x+1}$$

is continuous at $a = 3$.

$f(3) = \sqrt{4} = 2$

We want to show that $\lim\limits_{x \to 3} \sqrt{x+1} = 2$

Let $\varepsilon > 0$.

Note that $|\sqrt{x+1} - 2| = \left| \dfrac{(\sqrt{x+1} - 2)(\sqrt{x+1} + 2)}{(\sqrt{x+1} + 2)} \right| =$

$$= \left| \frac{x+1-4}{\sqrt{x+1}+2} \right| = \frac{|x-3|}{\sqrt{x+1}+2}.$$

Suppose $|x-3| < 1$

Then $-1 < x-3 < 1$. So, $2 < x < 4$. Then $3 < x+1 < 5$

Then $\sqrt{3} < \sqrt{x+1} < \sqrt{5}$. So, $\sqrt{3}+2 < \sqrt{x+1} + 2 < \sqrt{5} + 2$.

Thus, if $0 < |x-3| < 1$ then $\dfrac{|x-3|}{\sqrt{x+1}-2} < \dfrac{|x-3|}{\sqrt{3}+2}$.

Set $\delta = \min\{1, (\sqrt{3}+2)\varepsilon\}$.

If $0 < |x-3| < 1$ then

$$|\sqrt{x+1} - 2| = \frac{|x-3|}{\sqrt{x+1}+2} \underset{\substack{\uparrow \\ \boxed{|x-3|<1}}}{<} \frac{|x-3|}{\sqrt{3}+2} \underset{\substack{\uparrow \\ \boxed{|x-3| < \varepsilon(\sqrt{3}+2)}}}{<} \frac{1}{\sqrt{3}+2} \cdot \varepsilon(\sqrt{3}+2) = \varepsilon.$$

2. [10 points] Prove that

$$\lim_{x \to 100} \frac{1}{(x-100)^4}$$

does not exist. You may use theorems / homework statements from class, but you must prove that the above function satisfies any claims that you make about it.

We show that $\frac{1}{(X-100)^4}$ is unbounded near 100.

Let $M > 0$. Suppose $x \neq 100$.

Then

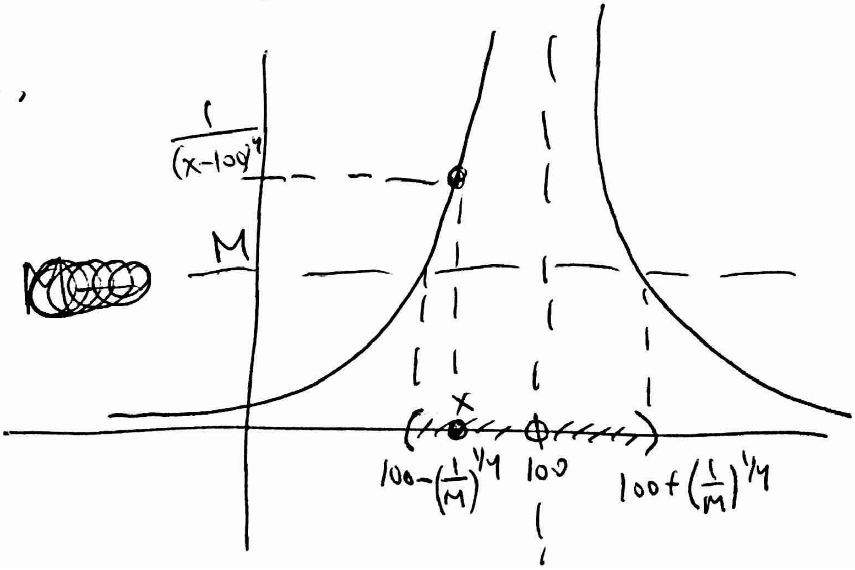
$\left|\frac{1}{(x-100)^4}\right| > M$ iff $\frac{1}{M} > |x-100|^4$

iff $\left(\frac{1}{M}\right)^{1/4} > |x-100|$.

Therefore, if $|x-100| < \left(\frac{1}{M}\right)^{1/4}$, then $\left|\frac{1}{(x-100)^4}\right| > M$.

By hw 3, 2(a), $\lim\limits_{x \to 100} \frac{1}{(x-100)^4}$ does not exist.

(or problem 3 on this test)



3. [10 points] Let $f : D \to \mathbb{R}$ be a function. Suppose that $\lim_{x\to a} f(x)$ exists for some $a \in \mathbb{R}$. Show that there exists $M > 0$ and $\delta > 0$ such that if $x \in D$ and $0 < |x - a| < \delta$ then $|f(x)| \leq M$.

See hw 3, 2(a).

4. [10 points - 5 each] True or False. If true, then prove it with a short proof. If false, then give an example showing that the statement can be false. You may use any definitions or theorems that we've learned in class or from homework for this problem.

(a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions. Suppose that the limit of f and the limit of $f + g$ both exist at $a \in \mathbb{R}$. Does this mean that the limit of g exists a ?

True

Since $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} [f(x)+g(x)]$ exist we know that $\lim\limits_{x\to a} [f(x)-(f(x)+g(x))]$ exists and equals $\lim\limits_{x\to a} [-g(x)]$. Hence

$-\lim\limits_{x\to a} (-g(x)) = \lim\limits_{x\to a} g(x)$ exists.

Easier: Since $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} [f(x)+g(x)]$ exist, so does [redacted] $\lim\limits_{x\to a} [f(x)+g(x)] - \lim\limits_{x\to a} [f(x)]$

$= \lim\limits_{x\to a} g(x).$

(b) Suppose that $f : D \to \mathbb{R}$ and a is a limit point of D. Suppose that $\lim\limits_{x\to a} |f(x)| = 0$. Then $\lim\limits_{x\to a} f(x) = 0$.

True

Let $\varepsilon > 0$. Since $\lim\limits_{x\to a} |f(x)| = 0$ there exists $\delta > 0$ so that if $0 < |x-a| < \delta$ then $\underbrace{||f(x)||}_{||f(x)|-0|} < \varepsilon$.

But $||f(x)|| = |f(x)|$.

So, if $0 < |x-a| < \delta$, then $\underbrace{|f(x)|}_{|f(x)-0|} < \varepsilon$.

Hence $\lim\limits_{x\to a} f(x) = 0$.

5. [10 points] Let D be a subset of the real numbers. Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R}$. Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ where $a \in \mathbb{R}$. Prove that $\lim_{x\to a} g(x) = L$.

You must only use definitions on this problem, no hw exercises or theorems from class.

Let $\varepsilon > 0$.

Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$ then $|f(x)-L| < \varepsilon$ or $-\varepsilon < f(x) - L < \varepsilon$.

Since $\lim_{x\to a} h(x) = L$, there exists $\delta_2 > 0$ so that if $0 < |x-a| < \delta_2$ then $-\varepsilon < h(x) - L < \varepsilon$. or $|h(x) - L| < \varepsilon$

Let $\delta = \min\{\delta_1, \delta_2\}$. then

If $0 < |x-a| < \delta$, then

$$-\varepsilon < f(x) - L \underset{\substack{\uparrow \\ \boxed{f(x)\leq g(x)}}}{\leq} g(x) - L \underset{\substack{\uparrow \\ \boxed{g(x)\leq h(x)}}}{\leq} h(x) - L < \varepsilon$$

that is, $-\varepsilon < g(x) - L < \varepsilon$

that is, $|g(x) - L| < \varepsilon$.